SISO T-S Fuzzy Systems as Universal Approximators

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Abstract. According to mathematical properties of continuous function, the universal approximation properties of fuzzy mapping in only one interval are analyzed, and some related theorems are proposed and proved. Then, the sufficient conditions of the general T-S fuzzy systems with universal approximation are derived from these theorems, which helps establish the new relationship between approximation error bounds and the number of the fuzzy rules of different T-S fuzzy systems. Finally, the paper proposes some suggestions to construct T-S fuzz systems with universal approximation.

Keywords: T-S fuzzy system, universal approximation, defuzzifier.

1. Introduction

The fuzzy systems can be classified into two major types, namely, Mamdani fuzzy systems and Takagi-Sugino (T-S) fuzzy systems. The primary difference between them lies in the fuzzy rule consequence. Mamdani fuzzy systems use fuzzy sets as rule consequence whereas T-S fuzzy systems employ linear function of input variables as rule consequence. For T-S fuzzy systems as universal approximators, there exist many results for existence conditions, sufficient conditions and necessary conditions. For sufficient conditions of T-S fuzzy systems as universal approximators, there exist some preferable results for T-S fuzzy system with linear consequence in the paper. But the results for the relations between the approximation accuracy and the membership functions, defuzzifiers and the number of fuzzy rule of the SISO T-S fuzzy systems for t-norm inference and implication engines are not convenient for applications.

2. Mathematical Principle

We consider the case where the desired function $g(x)$ is a continuous function on $U \subset R$. Before giving the approximation bounds, we define the ∞ norm for a bounded function $g(x)$ in U to be $\|g(x)\|_\infty = \sup_{x\in U}|g(x)|$ and introduce the modulus of continuity of $g(x)$ in $U \subset R$ to be

$$\eta(g,d,U) = \max_{x,y\in U;|x-y|\le d}|g(x)-g(y)| \tag{1}$$

Where d is an arbitrarily small positive number.

When $g(x)$ is continuously differentiable on U, it is easy to be verified by the differential mean theorem that

$$\eta(g,d,U) \le h\left\|\frac{\mathrm{d}\,g}{\mathrm{d}\,x_j}\right\|_\infty \tag{2}$$

Lemma 1. Suppose a continuous function $f(x)\ x\in[a,b],\ b>a$, it has the properties:

(1) $f(a) = f(b) = 0$

(2) $f(x)$ is almost-everywhere differentiable in the interval (a,b),

Then there holds

$$\|f(x)\|_\infty \le \|f'(x)\|_\infty\, {(b-a)}/{2} \tag{3}$$

Lemma 2: Suppose a continuous function $f(x)\ x\in[a,b],\ b>a$, it has the properties:

(1) $f(a) = f(b) = 0$

(2) $f(x)$ is almost-everywhere twice continuously differentiable in the interval (a,b).

Then there holds

$$\|f(x)\|_\infty \le \|f''(x)\|_\infty {(b-a)^2}/{8} \tag{4}$$

Lemma 3: Suppose a continuous function $f(x)$ $x \in [a,b], b > a$, it has the properties:

(1) $f(a) = f(b) = 0$, $f'(a) = f'(b) = 0$;

(2) $f(x)$ is almost-everywhere twice continuously differentiable in the interval (a,b).

Then there holds

$$\|f(x)\|_\infty \le \|f''(x)\|_\infty {(b-a)^2}/{16} \tag{5}$$

Lemma 4: Suppose a continuous function $f(x)$ $x \in [a,b]$, $b > a$, it has the properties:

(1) $f(a) = f(b) = 0$, $f'(a) = f'(b) = 0$;

(2) $f(x)$ is almost-everywhere thrice continuously differentiable in the interval (a,b),

Then there holds

$$\|f(x)\|_\infty \le \|f'''(x)\|_\infty {(b-a)^3}/{192} \tag{6}$$

3. The SISO Fuzzy Mapping on One Interval

Definition1.A function can be called a half-part membership function of fuzzy systems, when it has such properties as follows:

(1) $A(x)$ is continuous functions in the interval $[a,b]$;

(2) $0 \le A(x) \le 1$ in the interval and at least there holds $A(x_0) = 1$ where $x_0 = a$ or $x_0 = b$;

(3) $A(x) = 0, \quad \forall\ x \notin [a,b]$.

Definition 2. A function defined by the definition 1 can be called the right-part membership function, denoted by $A^R(x)$, if it has properties as follows

(1) $A(a) = 1$ and $A(b) = 0$.

Definition 3. A function defined by the definition 1 can be called the left-part membership function, denoted by $A^L(x)$, if it has properties as follows

(1) $A(a) = 0$ and $A(b) = 1$.

The fuzzy rule of the T-S fuzzy mapping can be written as

R_i : IF x is A_i, THEN $y_i = h_i(x)\ (i = 0,1)$.

Where $A_0(x)$ is composed of only the right-part membership function $A_0^R(x)$ and $A_1(x)$ is composed of only the left-part membership function $A_1^L(x)$, and $A_0(x) + A_1(x) \ne 0$ in the interval $[0,1]$.

The T-norm operators are used as the fuzzy inference and implication engine.

The deffuzification can use any type of deffuzifiers (e.g., center-average deffuzifier, maximum defuzzifier).

For this example the linear T-S fuzzy mapping can be written as follows:

$$f(x) = D_0(x,y)h_0(x) + D_1(x,y)h_1(x) \tag{7}$$

$$|f(x) - g(x)| = |D_0(x,y)h_0(x) + D_1(x,y)h_1(x) - g(x)| \tag{8}$$

Where

$$D_0(x,y) = \frac{(R_0(x,y))^\alpha}{(R_0(x,y))^\alpha + (R_1(x,y))^\alpha},$$

$$D_1(x,y)=\frac{(R_1(x,y))^\alpha}{(R_0(x,y))^\alpha+(R_1(x,y))^\alpha}$$
$$R_i(x,y)=A_i(x)*B_i(y)\ (i=0,1)$$

Proposition 1: The error between any T-S fuzzy mapping $f(x)$ defined by equation (8) and any differentiable function $g(x)$ satisfies

$$|f(x)-g(x)|\le\eta \tag{9}$$

Where η is defined similarly as equation (1) and

$$\eta\le|x_1-x_0|\left\|\frac{\mathrm{d}(f-g)}{\mathrm{d}x}\right\|_\infty . \tag{10}$$

SISO T-S FUZZY SYSTEM

The fuzzy rule of the T-S fuzzy mapping can be written as

$$R_i:\text{IF}\quad x\quad\text{is}\quad A_i,\ \text{THEN}\quad y=a_i+b_ix \tag{11}$$

The deffuzification can use any types of deffuzifiers (e.g., center-average deffuzifier, maximum defuzzifier).

So the function of the fuzzy system, which uses equation (14) as its rule base can be written as follows:

$$f_T(x)=\sum_{i=2}^{n}\left(D_{i-1}(x)(a_{i-1}+b_{i-1}x)+D_i(x)(a_i+b_ix)\right) \tag{12}$$

Where

$$D_{i-1}(x)=\frac{(R_{i-1}(x))^\alpha}{(R_{i-1}(x))^\alpha+(R_i(x))^\alpha}\ ,$$
$$D_i(x)=\frac{(R_i(x))^\alpha}{(R_{i-1}(x))^\alpha+(R_i(x))^\alpha}\ ,$$
$$\sum_{i=1}^{n}\left(D_{i-1}(x)+D_i(x)\right)=1$$
$$\alpha\in[0,+\infty]$$

$R_i(x)\quad(i=1,2,\cdots,n-1)$ is composed of $R_i^L(x)$ and $R_i^R(x)$, $R_i^L(x)$ is defined in the interval $[x_{i-1},x_i]$ and $R_i^R(x)$ is defined in the interval $[x_i,x_{i+1}]$, $R_0(x)$ is composed of $R_0^R(x)\ R_0^R(x)$ defined in the interval $[x_0,x_1]$ and $R_n(x)$ is composed of $R_n^L(x)$ defined in the interval $[x_{n-1}x_n]$.

$$f(x)=\sum_{i=1}^{n}f_i(x) \tag{13}$$

Where

$$f_i(x)=D_{i-1}(x)\left(g(x_{i-1})+g'(x_{i-1})(x-x_{i-1})\right)$$
$$+D_i(x)\left(g(x_i)+g'(x_i)(x-x_i)\right)$$
$$x\in[x_{i-1},x_i](i=1,2,\cdots,n).$$

So if we know the properties of every piece-patch fuzzy mapping, then we can know the fuzzy system composed of all piece-patch fuzzy mappings whose interval is continuous.

Proposition 2: Suppose that n_0 equally distributed fuzzy mappings are assigned to constitute the SISO T-S fuzzy systems (17) and the functions $D_{i-1}(x),D_i(x)\quad(i=1,2,\cdots,n)$ are continuous and almost-everywhere continuously differentiable and if $g(x)$ is continuously differentiable on $[a,b]\quad(a<b)$ and $f_T(x_i)=g(x_i)\ i=1,2,\cdots,n$, then

$$\|f_T(x)-g(x)\|_\infty\le\|g'(x)\|_\infty{(b-a)}/{2n_0} \tag{14}$$

Theorem 4: Suppose that n_0 equally distributed fuzzy mappings are assigned to constitute the SISO T-S fuzzy systems (13) and the functions $D_{i-1}(x),D_i(x)\quad(i=1,2,\cdots,n)$ are continuous and almost-everywhere twice continuously differentiable, then for any given polynomial $P_d(x)$ defined on $C[0,1]$ and approximation error bound $\varepsilon>0$, there exists a T-S fuzzy system such that $\|f_T(x)-P_d(x)\|<\varepsilon$ holds when

$$n_0 > \frac{1}{4}\sqrt{\frac{1}{\varepsilon}\cdot\left\|\frac{\mathrm{d}^2(P_d - f_T)}{\mathrm{d}x^2}\right\|_\infty} \tag{15}$$

Corollary 1: Suppose that n_0 equally distributed fuzzy mappings are assigned to constitute the SISO T-S fuzzy systems (17) and the functions $D_{i-1}(x), D_i(x)$ $(i = 1,2,\cdots,n)$ are triangle-shaped or trapezoid-shaped or subsection linear functions, then for any given polynomial $P_d(x)$ defined on $C[0,1]$ and approximation error bound $\varepsilon > 0$, there exists a T-S fuzzy system such that $\|f_T(x) - P_d(x)\| < \varepsilon$ holds when

$$n_0 > \frac{1}{4}\sqrt{\frac{1}{\varepsilon}\cdot\left\|\frac{\mathrm{d}^2 P_d}{\mathrm{d}x^2}\right\|_\infty} \tag{16}$$

Theorem 5: Suppose that n_0 equally distributed fuzzy mappings are assigned to constitute the SISO T-S fuzzy systems (17) and the functions $D_{i-1}(x), D_i(x)$ $(i = 1,2,\cdots,n)$ are continuous and almost-everywhere twice continuously differentiable, then for any real continuous function $g(x)$ defined on $C[0,1]$ and approximation error bound $\varepsilon > 0$, there exists a T-S fuzzy system such that $\|f_T(x) - g(x)\| < \varepsilon$ holds when

$$n_0 > \frac{1}{4}\sqrt{\frac{1}{\varepsilon - \varepsilon_1}\cdot\left\|\frac{\mathrm{d}^2(P_d - f_T)}{\mathrm{d}x^2}\right\|_\infty} \tag{17}$$

Where $0 < \varepsilon_1 < \varepsilon$ and $\|g(x) - P_d(x)\| < \varepsilon_1$ holds.

Corollary 2: Suppose that n_0 equally distributed fuzzy mappings are assigned to constitute the SISO T-S fuzzy systems (13) and the functions $D_{i-1}(x), D_i(x)$ $(i = 1,2,\cdots,n)$ are triangle-shaped, trapezoid-shaped or subsection linear functions, then for any given real continuous function $g(x)$ defined on $C[0,1]$ and approximation error bound $\varepsilon > 0$, there exists a T-S fuzzy system such that $\|f_T(x) - g(x)\| < \varepsilon$ if

$$n_0 > \frac{1}{4}\sqrt{\frac{1}{\varepsilon - \varepsilon_1}\cdot\left\|\frac{\mathrm{d}^2 P_d}{\mathrm{d}x^2}\right\|_\infty} \tag{18}$$

Where $0 < \varepsilon_1 < \varepsilon$ and $\|g(x) - P_d(x)\| < \varepsilon_1$ holds.

Theorem 6: Suppose that n_0 equally distributed fuzzy mappings are assigned to constitute the SISO T-S fuzzy systems (17) and the functions $D_{i-1}(x), D_i(x)$ $(i = 1,2,\cdots,n)$ are continuous and almost-everywhere thrice continuously differentiable and $D_o'(x,y) = 0, D_1'(x,y) = 0$, then for any given polynomial $P_d(x)$ defined on $C[0,1]$ and approximation error bound $\varepsilon > 0$, there exists a T-S fuzzy system such that $\|f_T(x) - P_d(x)\| < \varepsilon$ holds when

$$n_0 > \frac{1}{4}\sqrt[3]{\frac{1}{3\varepsilon}\left\|\frac{\mathrm{d}^3(P_d - f_T)}{\mathrm{d}x^3}\right\|_\infty} \tag{19}$$

Corollary 3: Suppose that n_0 equally distributed fuzzy mappings are assigned to constitute the SISO T-S fuzzy systems (17) and $D_{i-1}(x), D_i(x)$ $(i = 1,2,\cdots,n)$ of fuzzy system is trapezoid-shaped, and if $g(x)$ is twice continuously differentiable on $C^2[0,1]$ and $f_T(x_i) = P_d(x_i)$ $(i = 1,2,\cdots,n)$, then exists a T-S fuzzy system such that $\|f_T(x) - P_d(x)\| < \varepsilon$ holds when

$$n_0 > \frac{1}{4}\sqrt[3]{\frac{1}{3\varepsilon}\left\|\frac{\mathrm{d}^3 P_d}{\mathrm{d}x^3}\right\|} \tag{20}$$

Theorem 7: Suppose that n_0 equally distributed fuzzy mappings are assigned to constitute the SISO T-S fuzzy systems (17) and the functions $D_{i-1}(x), D_i(x)$ $(i = 1,2,\cdots,n)$ are continuous and almost-everywhere thrice continuously differentiable and $D_i'(x_i) = 0, D_i'(x_j) = 0 (i \neq j)$, then for any real continuous function $g(x)$ defined on $C[0,1]$ and approximation error bound $\varepsilon > 0$, there exists a T-S fuzzy system such that $\|f_T(x) - g(x)\| < \varepsilon$ holds when

$$n_0 > \frac{1}{4}\sqrt[3]{\frac{1}{3\varepsilon - \varepsilon_1}\left\|\frac{\mathrm{d}^3(P_d - f_T)}{\mathrm{d}x^3}\right\|} \tag{21}$$

Where $0 < \varepsilon_1 < \varepsilon$ and $\|g(x) - P_d(x)\| < \varepsilon_1$ holds.

Corollary 4: Suppose that n_0 equally distributed fuzzy mappings are assigned to constitute the SISO T-S fuzzy systems (17) and the functions $D_{i-1}(x), D_i(x)$ $(i=1,2,\cdots,n)$ are trapezoid-shaped, then for any given real continuous function $g(x)$ defined on $C[0,1]$ and approximation error bound $\varepsilon>0$, there exists a T-S fuzzy system such that $\|f_T(x)-g(x)\|<\varepsilon$ if

$$n_0 > \frac{1}{4}\sqrt[3]{\frac{1}{3\varepsilon-\varepsilon_1}\left\|\frac{\mathrm{d}^3 P_d}{\mathrm{d}x^3}\right\|} \tag{22}$$

Where $0<\varepsilon_1<\varepsilon$ and $\|g(x)-P_d(x)\|<\varepsilon_1$ holds.

Theorem 8: For any given real continuous and constant differential function $g(x)$ defined on $C[0,1]$, there exists a T-S fuzzy system (17) such that

$$f_T(x)=g(x) \tag{23}$$

Theorem 9: Suppose that n_0 equally distributed fuzzy mappings are assigned to constitute the SISO T-S fuzzy systems (13) and $D_{i-1}(x), D_i(x)$ $(i=1,2,\cdots,n)$ are constant twice differential, then For any given real continuous function $g(x)$ with constant twice differential defined on $C[0,1]$, there exists a T-S fuzzy system such that

$$f_T(x)=g(x) \tag{24}$$

4. Conclusion

In this study, from the mapping point of view, the universal approximation error bound for general SISO T-S fuzzy systems and some special SISO T-S fuzzy systems were discussed. According to the interpolation mechanism of fuzzy system and the property of the continuous function, many results were obtained. Based on different defuzzifiers, many different results were presented.

1. the $\alpha=0$ case, for linear consequence T-S fuzzy systems, the fuzzy function is discontinuous at the points $x=x_i$, at least. Because of this property, the SISO T-S fuzzy system with such defuzzifier is of poor approximation in the neighborhood of the point $x=x_i$.

2. the $\alpha=+\infty$ case for linear consequence T-S fuzzy systems, the fuzzy function is discontinuous at points $A_{i-1}(x)=A_i(x)$ (where $A_i(x)$ is the membership function of the input of the fuzzy system). Because of this property, the SISO T-S fuzzy system with such defuzzifier is of poor approximation in the neighborhood of the point $A_{i-1}(x)=A_i(x)$.

3. the $\alpha=(0,+\infty)$ case. The center-average defuzzifier can replace all other defuzzifiers according to the section of different membership functions. This case has proper continuous property. In order to approximation a continuous function, by SISO T-S fuzzy system, this case is the best choice.

In the first two cases, the choice of the membership function of the fuzzy systems should not be much complicated because of the exceedingly poor property of the defuzzifiers. In the last case, the SISO T-S fuzzy systems are the best approximators to approximate general continuous function.

Among all continuous subsection linear membership functions, the approximation conditions of the SISO T-S fuzzy systems with the triangle-shaped, Trapezoid-shaped or subsection linear membership function were presented alone. In practice, we can choose different defuzzifier, membership function and inference according to the desired function and properties of these fuzzy systems.

References

[1] H.L. Liu, R.P. Feng: New Fuzzy Distance Definitions of Fuzzy Numbers, Fuzzy Systems and Mathematics, Vol. 19 (2005), No.2, p.106-109.

[2] C.P. Kwong: The Minimum worst case error of fuzzy approximators, IEEE Trans. SMC, B, Vol. 31 (2001), No. 6, p.714-716.

[3] K. Zeng, N.Y. Zhang, W.L. Xu: A comparative study on sufficient conditions for Takagi-Sugeno fuzzy systems as universal approximators, IEEE Trans. Fuzzy Syst, Vol. 8 (2000), p.773-780.

[4] L.X. Wang, C. Wei: Approximation accuracy of some neuro-fuzzy approaches, IEEE Trans. Fuzzy Syst, Vol. 8 (2000), No. 4, p.470-478.

[5] W. T. Chen: Sufficient conditions on general fuzzy systems as function approximators, IEEE Trans. SMC, B, Vol. 3 (2001) No. 2, p.270-174.

[6] M. Landajo, M. J. Rio, and R. Perez: A note on smooth approximation capabilities of fuzzy systems, IEEE Trans. Fuzzy Syst., Vol. 9 (2001) No. 2, p. 229-237.

[7] S. Mitaim, B. Kosko: The shape fuzzy sets in adaptive function approximation, IEEE Trans. Fuzzy Syst., Vol. 9 (2001) No. 4, p. 637-656.

[8] H.Han, C.Y. Su, Y. Stepanenko: Adaptive control of a class of nonlinear systems with nonlinearly parameterized fuzzy approximators, IEEE Trans. Fuzzy Syst., Vol. 9 (2001), No. 2, p. 315-323.