Global existence and nonexistence to Parabolic Systems with Different Type Nonlinearities

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Abstract. This paper deals with the existence and nonexistence of global positive solutions of diffusion systems with nonlinear terms. Sufficient and necessary conditions on the global existence of the positive solution are obtained.

Keywords: Nonlinear parabolic equations; nonlinear boundary flux; global solution; finite time blow-up.

1. Introduction

In this paper, we consider the following parabolic equations with different type nonlinearities. The main questions we here address are the global existence and the nonexistence solutions.

$$\begin{cases} u_t = u + f(u,v), & v_t = v + g(u,v), & x \in \Omega, & t > 0, \\ \frac{u}{n} = f_1(u,v), & \frac{v}{n} = g_1(u,v), & x \in \Omega & t > 0, \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), & x \in \bar{\Omega}. \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundaryΩ, $u_0(x)$ and $v_0(x)$ are positive smooth functions satisfying the compatibility conditions, n is the unit outward normal vector. I will give the examples in section (2).

Recently, many important results have appeared on blow-up problems for nonlinear parabolic systems. Some of those results are stated below. In [1] Escobedo et al studied a parabolic system with coupled inner sources of the form

$$u_t = u + v^p, \quad v_t = v + u^p \quad x \in \Omega\ , t > 0$$

With $p, q > 1$, $\Omega \subset \mathbb{R}^N$, system (2) is with Dirichlet boundary conditions. the authors showed that for$pq \leq 1$, every solution of systems (2) is global, while for $pq > 1$ there are solutions that blow up and others that are global.

Xianfa Song [2] studied the single equation

$$\begin{cases} u_t = \Delta u + u^\alpha, & x \in \Omega, & t > 0 \\ \frac{u}{n} = u^\beta, & x \in \partial\Omega, & t > 0 \\ u(x,0) = u_0(x), & x \in \bar{\Omega}. \end{cases} \tag{1.2}$$

Where, $\alpha > 0$ and $\beta > 0$, they conclude the sufficient and necessary conditions on the global existence of system (3) is that $\alpha \leq 1$ and $\beta \leq 1$.

Currently, the sufficient and necessary conditions on the global existence were obtained by Keng Deng[3] for heat equations coupled via boundary flux

$$\begin{cases} u_t = u, & v_t = v, & x \in \Omega, & t > 0 \\ \frac{u}{n} = v^p, & \frac{v}{n} = u^q, & x \in \partial\Omega, & t > 0 \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), & x \in \bar{\Omega}. \end{cases} \tag{1.3}$$

Ω is bounded in $\mathbb{R}^N$, we obtained the necessary and sufficient conditions on blow-up in finite time of system (4) is $pq > 1$.

This paper, we consider the systems with exponent and power types nonlinear terms. In section 2, we will give several types example and considering results. In section 3, we will proof these results.

2. Main results

Now we state the examples and the main results of this paper.

I: Source-source or flux-flux coupled

2.1 Coupled power types

Case (1): $f(u,v)=v^p$, $g(u,v)=\mu^q$, $f_1(u,v)=e^{\alpha\mu}$, $g_1(u,v)=e^{\beta v}$

Case (2): $f(u,v)=e^{\alpha\mu}$, $g(u,v)=e^{\beta v}$, $f_1(u,v)=v^p$, $g_1(u,v)=\mu^q$

Where $p,q\geq 0$; α,β are constants.

Theorem 2.1: Assume case (1) or (2), then the solution of (1) exists globally if and only if

$$\alpha\leq 0,\ \beta\leq 0 \text{ and } pq\leq 1. \tag{i}$$

2.2 Coupled exponent types

Case (3): $f(u,v)=e^{\alpha v}$, $g(u,v)=e^{\beta\mu}$, $f_1(u,v)=\mu^p$, $g_1(u,v)=v^q$

Case (4): $f(u,v)=\mu^p$, $g(u,v)=v^q$, $f_1(u,v)=e^{\alpha v}$, $g_1(u,v)=e^{\beta\mu}$

where $\alpha,\beta\geq 0$, p and q are constants.

Theorem 2.2: Assume case (3) or (4), then the solution of (1) exists globally if and only if

$$\alpha=0 \text{ or } \beta=0,\ p\leq 1 \text{ and } q\leq 1. \tag{ii}$$

II: Source-flux coupled

2.3 Coupled power types

Case (i): $f(u,v)=v^q$, $g(u,v)=e^{\alpha v}$, $f_1(u,v)=e^{\beta\mu}$, $g_1(u,v)=\mu^q$

Case (ii): $f(u,v)=e^{\alpha\mu}$, $g(u,v)=\mu^q$, $f_1(u,v)=v^p$, $g_1(u,v)=e^{\beta v}$

where $p,q\geq 0$; α,β are constants.

Theorem 2.3: Assume case (i) or (ii), then the solution of (1) exists globally if and only if

$$\alpha\leq 0,\ \beta\leq 0 \text{ and } pq\leq 1. \tag{iii}$$

3. Proof of Theorem (2.1-2.3)

3.1. Proof of Theorem 2.1

If **case (1)** holds, then (1) transforms

$$\begin{cases} u_t=u+v^p, & v_t=v+u^q, & x\in\Omega\ ,t>0,\\ \frac{u}{n}=e^u, & \frac{v}{n}=e^{\beta v}, & x\in\Omega\ ,t>0,\\ u(x,0)=u_0(x), & v(x,0)=v_0(x), & x\in\bar{\Omega}. \end{cases} \tag{3.1}$$

The first, we prove the sufficiency of theorem 2.1. That is to say, if condition (i) satisfies, we will prove that the solution (u,v) of(3.1)exists globally.

Proof. Consider

$$\begin{cases} \overline{u}_t=\overline{u}+\overline{v}^p, & \overline{v}_t=\overline{v}+\overline{u}^q, & x\in\Omega\ ,t>0,\\ \frac{\overline{u}}{n}=1, & \frac{\overline{v}}{n}=1, & x\in\Omega\ ,t>0,\\ \overline{u}(x,0)=u_0(x), & \overline{v}(x,0)=v_0(x), & x\in\bar{\Omega}. \end{cases} \tag{3.2}$$

Obviously, the solution $(\overline{u},\overline{v})$ of (2.2) is upper solution of (3.1). By [4], we know if $pq\leq 1$ satisfied, then the solution $(\overline{u},\overline{v})$ of (3.2) exists globally. So the solution (u,v) of (3.1) exists globally.

The proof of sufficiency is complete.

Next, we will complete the proof of the necessary by the following series of lemmas.

Lemma 3.1 If $\alpha>0$ or $\beta>0$, then the solution $(u,v)$of (6) blows up in finite time.

Proof. As $\alpha>0$, consider

$$\begin{cases} \underline{u}_t=\underline{u}, & \underline{v}_t=\underline{v}, & x\in\Omega\ ,t>0,\\ \frac{\underline{u}}{n}=e^{\alpha\underline{u}}, & \frac{\overline{v}}{n}=0, & x\in\Omega\ ,t>0,\\ \underline{u}(x,0)=u_0(x), & \underline{v}(x,0)=v_0(x), & x\in\bar{\Omega}. \end{cases} \tag{3.3}$$

compare (8) with (3.1), by the comparison principle, $(\underline{u},\underline{v})$ is a sub-solution. Obviously, when $\alpha>0$, $\underline{u}$ blows up in finite time. Consequently, the solution (u,v) of (3.1) blows up in finite time.

When $\beta>0$, the proof is similar.

Lemma 3.2 If $pq>1$, then the solution (u,v) of (3.1) blows up in finite time.

Proof. Consider

$$\begin{cases} \underline{u}_t = \underline{u} + \underline{v}^p, & \underline{v}_t = \underline{v} + \underline{u}^q, & x \in \Omega\ , t > 0, \\ \frac{\underline{u}}{n} = 0, & \frac{\underline{v}}{n} = 0, & x \in \Omega\ , t > 0, \\ \underline{u}(x,0) = u_0(x), & \underline{v}(x,0) = v_0(x), & x \in \bar{\Omega}, \end{cases} \tag{3.4}$$

compare (3.4) with (3.1), $(\underline{u}, \underline{v})$ is a sub-solution of (2.1). we know if $pq > 1$, $(\underline{u}, \underline{v})$ blows up in finite time. Accordingly, the solution (u, v) of (3.1) blows up in finite time.

The proof that condition (i) is necessary for the existence of a global solution follows from the two lemmas.

When case (2), the proof is parallel to case (1).

3.2. Proof of theorem 2.2

If case (3), equations (1) transforms

$$\begin{cases} u_t = u + e^{\alpha v}, & v_t = v + e^{\beta u}, & x \in \Omega\ , t > 0, \\ \frac{u}{n} = u^p, & \frac{v}{n} = v^q, & x \in \Omega\ , t > 0, \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), & x \in \bar{\Omega}. \end{cases} \tag{3.5}$$

We will prove several lemmas to prove theorem 2.2.

Lemma 3.3 If $\alpha = 0$ or $\beta = 0$ and $p \le 1$ and $q \le 1$, then the solution (u, v) of (2.5) exists globally.

Proof. Consider

$$\begin{cases} u_t = u + 1, & v_t = v + e^{\beta u}, & x \in \Omega\ , t > 0, \\ \frac{u}{n} = u^p, & \frac{v}{n} = v^q, & x \in \Omega\ , t > 0, \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), & x \in \bar{\Omega}. \end{cases} \tag{3.6}$$

Obviously, when $p < 1$, u exists globally. The following, we prove that v exists globally. We suppose v blows up in finite time T. In $[0, T]$, u reaches maximum, denotes M. Then, consider

$$\begin{cases} \overline{v}_t = \overline{v} + e^{\beta M}, & (x,t) \in \Omega \times (0,T), \\ \frac{\overline{v}}{n} = \overline{v}^q, & (x,t) \in \Omega \times (0,T), \\ \overline{v}(x,0) = v_0(x), & x \in \bar{\Omega}. \end{cases} \tag{3.7}$$

By comparison principle, we know $\overline{v}$ is a super solution of v. Since$q \le 1$, we can show that $\overline{v}$ is global. Accordingly, v is global. It is a contradiction with supposition.

So the solution (u, v) of (2.5) exists globally.

When $p \le 1$, $q \le 1$ and$=0$, the proof is parallel.

Lemma 3.4 If $p > 1$ or $q > 1$, then the solution (u, v) of (10) blows up in finite time.

Proof. Consider

$$\begin{cases} \underline{u}_t = \underline{u}, & \underline{v}_t = \underline{v}, & x \in \Omega\ , t > 0, \\ \frac{\underline{u}}{n} = \underline{u}^p, & \frac{\underline{v}}{n} = 0, & x \in \Omega\ , t > 0, \\ \underline{u}(x,0) = u_0(x), & \underline{v}(x,0) = v_0(x), & x \in \bar{\Omega}. \end{cases} \tag{3.8}$$

By comparison principle, $(\underline{u},\underline{v})$ is sub-solution. when$p > 1$, $\underline{u}$ blows up in finite time. Therefore, the solution (u, v) of (10) blows up in finite time.

When $q > 1$, the proof is similar.

Lemma 3.5 Assume $\alpha > 0$, $\beta > 0$, then the solution of (3.5) blows up in finite time.

Proof. Consider

$$\begin{cases} \underline{u}_t = \underline{u} + e^{\alpha \underline{v}}, & \underline{v}_t = \underline{v} + e^{\beta \underline{u}}, & x \in \Omega\ , t > 0, \\ \frac{\underline{u}}{n} = 0, & \frac{\underline{v}}{n} = 0, & x \in \Omega\ , t > 0, \\ \underline{u}(x,0) = u_0(x), & \underline{v}(x,0) = v_0(x), & x \in \bar{\Omega}. \end{cases} \tag{3.9}$$

In [1], we know that if $\alpha > 0$ and $\beta > 0$, then $(\underline{u}, \underline{v})$ blows up in finite time. Compare (3.9) with (3.5), $(\underline{u}, \underline{v})$ is sub-solution of (3.5). So (u, v) blows up in finite time.

When case (4), the proof is similar.

The proof of theorem 2.2 is complete follows from lemma 3.3---3.5.

3.3 Proof of theorem 2.3

If **case(i)**, equation (1.1) transforms

$$\begin{cases} u_t = u + v^p, & v_t = v + e^{\alpha v}, & x \in \Omega\ , t > 0, \\ \frac{u}{n} = e^{\beta u}, & \frac{v}{n} = u^q, & x \in \Omega\ , t > 0, \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), & x \in ^{-}. \end{cases} \tag{3.10}$$

To prove theorem 2.3, we need prove several lemmas.

Lemma 3.6 If $\alpha \le 0$ or $\beta \le 0$ and $pq \le 1$, then the solution (u, v) of (3.10) exists globally.

Proof. Consider

$$\begin{cases} \overline{u}_t = \underline{u} + \overline{v}^p, & \overline{v}_t = \overline{v} + 1, & x \in \Omega\ , t > 0, \\ \frac{\overline{u}}{n} = 1, & \frac{\overline{v}}{n} = \overline{u}^q, & x \in \Omega\ , t > 0, \\ \overline{u}(x,0) = u_0(x), & \overline{v}(x,0) = v_0(x), & x \in \bar{\Omega}. \end{cases} \tag{3.11}$$

In [2], Sining Zheng and Xianfa Song studied, when $pq \le 1$, the solution of (3.11) is global. Compare (3.11) with (3.10), $(\underline{u}, \underline{v})$ is super solution of (3.10). So, the solution (u, v) of (3.10) is global.

Lemma 3.7 If $\alpha > 0$ or $\beta > 0$, then the solution (u, v) of (3.10) blows up in finite time.

Proof. Consider

$$\begin{cases} w_t = w + e^{\alpha w}, & x \in \Omega\ , t > 0, \\ \frac{w}{n} = 0, & x \in \Omega\ , t > 0, \\ w(x,0) = u_0(x), & x \in \bar{\Omega}. \end{cases} \tag{3.12}$$

Obviously, when $\alpha > 0$, w blows up in finite time. Set $\underline{u} = 0, \underline{v} = w$, by comparison principal, $(\underline{u}, \underline{v})$ is a sub-solution. It is obvious that the solution (u, v) blows up in finite time.

When $\beta > 0$, the proof is similar.

Lemma 3.8 If $pq > 1$, then the solution (u, v) of (3.10) blows up in finite time.

Before giving the proof, we introduce the function $\varphi_0(x)$, which be the first eigenfunction of

$$\Delta\varphi + \lambda\varphi = 0 \quad \text{in } \Omega, \qquad \varphi = 0 \quad \text{on } \partial\Omega$$

with the first eigenvalue λ_0, normalized by $\|\varphi_0\|_\infty = \max_{\bar{\Omega}} \varphi_0(\cdot) = 1$. Then $\varphi_0 > 0$ in Ω and

$$c_1 \le \left|\frac{\partial\varphi_0}{\partial\eta}\right|_{\partial\Omega} = \left(-\frac{\partial\varphi_0}{\partial\eta}\right)\bigg|_{\partial\Omega} \le c_2$$

for some constants $c_1, c_2 > 0$. Moreover, there exist positive constants ε_0 and $c_3$such that

$$|\nabla\varphi_0| \ge c_1/2 \quad \text{for } x \in \Omega_1 = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \le \varepsilon_0\},$$
$$\varphi_0 \ge c_3 \qquad \text{for } x \in \Omega_2 = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \ge \varepsilon_0\}$$

Denote

$$\max_{\bar{\Omega}} |\nabla\varphi_0| = c_4 \ge c_2.$$

Proof. Consider

$$\begin{cases} w_t = w + z^p, & z_t = z, & x \in \Omega\ , t > 0, \\ \frac{w}{n} = 0, & \frac{z}{n} = w^q, & x \in \Omega\ , t > 0, \\ w(x,0) = u_0(x), & z(x,0) = v_0(x), & x \in \bar{\Omega}. \end{cases} \tag{3.13}$$

Inspired by [3], construct

$$\underline{w} = \delta[(1-ct)^2 + a^2\phi^2]^{-k}, \quad \underline{z} = \delta[(1-ct) + a\phi]^{-l}, \quad x \in \Omega,\ t \in (0, \frac{1}{c}),$$

where $\phi = \phi_0$, $\delta = \min(u_0(x), v_0(x))$, $k = \frac{1+\frac{p}{2}}{pq-1}, l = \frac{1+2q}{pq-1}$, $a, c > 0$ to be determined. A simple computation shows

$$\underline{w}_t \le 2kc\delta[(1-ct)^2 + a^2\phi^2]^{-k-1}$$
$$\underline{z}_t \le lc\delta[(1-ct) + a\phi]^{-l-1}$$
$$\begin{aligned} \Delta\underline{w} = {} & 2k(k+1)\delta a^4[(1-ct)^2 + a^2\phi^2]^{-k-2}\phi^2(-\frac{\partial\phi}{\partial n})^2 + \\ & 2k\delta a^2[(1-ct)^2 + a^2\phi^2]^{-k-1}\phi(-\phi) - 2k\delta a^2[(1-ct)^2 + a^2\phi^2]^{-k-1}(\frac{\partial\phi}{\partial n})^2 \\ \ge {} & -2k\delta a^2 c_4^2[(1-ct)^2 + a^2\phi^2]^{-k-1}, \end{aligned}$$
$$\Delta\underline{z} = l\delta a[(1-ct) + a\phi]^{-l-1}(-\phi) + l(l+1)\delta a^2[(1-ct) + a\phi]^{-l-2}(\frac{\partial\phi}{\partial n})^2,$$
$$\frac{\partial\underline{w}}{\partial n} = 2k\delta a^2[(1-ct)^2 + a^2\phi^2]^{-k-1}\phi(-\frac{\partial\phi}{\partial n}),$$

$$\frac{\partial \underline{z}}{\partial n} = l\delta a[(1-ct)+a\phi]^{-l-1}(-\frac{\partial \phi}{\partial n})$$
$$\underline{w}^q = \delta^q[(1-ct)^2 + a^2\phi^2(x)]^{-kq}$$
$$\underline{z}^p = \delta^p[(1-ct)+a\phi(x)]^{-lp} \geq 2^{-\frac{lp}{2}}\delta^p[(1-ct)^2 + a^2\phi(x)]^{-\frac{lp}{2}}.$$

If $x \in \Omega_1$, then

$$\Delta \underline{z} \geq \frac{c_1^2 l(l+1)a^2[(1-ct)+a\phi]^{-l-2}}{4}.$$

If $x \in \Omega_2$, then

$$\Delta \underline{z} \geq l\delta\lambda_0 c_3 a[(1-ct)+a\phi]^{-l-1}.$$

Since $k = \frac{1+\frac{p}{2}}{pq-1}$, $l = \frac{1+2q}{pq-1}$, we have $2(k+1) = lp, 2kq = l+1$. If $a = \min(\frac{\delta^{\frac{p-1}{2}}}{2^{\frac{k+3}{2}}k^{\frac{1}{2}}c_4}, \frac{\delta^q - 1}{2^k + 3k})$, $c = \min(\frac{\delta^p - 1}{2^{k+3}k}, \frac{a^2(l+1)c_1^2}{4}, a\lambda_1 c_3)$, we get

$$\underline{w}_t \leq \Delta \underline{w} + \underline{z}^p, \quad \underline{z}_t \leq \Delta \underline{z}.$$
$$\frac{\partial \underline{w}}{\partial n}|_{\partial\Omega} = 0, \qquad \frac{\partial \underline{z}}{\partial n}|_{\partial\Omega} \leq \underline{w}^q.$$
$$\underline{w}(x,o) = \delta[1+a^2\phi^2(x)]^{-k} \leq \delta \leq u_0(x), \quad \underline{z}(x,o) = \delta[1+a\phi(x)]^{-l} \leq \delta \leq v_0(x).$$

It shows that $(\underline{w}, \underline{z})$ is a sub-solution of (3.13). When $pq > 1$, $(\underline{w}, \underline{z})$ blows up in finite time on the boundary. Accordingly, (w, z) blows up in finite time. Compare (3.13) with (3.10), (w, z) is a sub-solution of (3.10). So we get (u, v) blows up in finite time. The proof of theorem 2.3 is completed by lemmas 3.6--3.8 when case (i).

When case (ii), the proof is parallel to case (i).

4. Conclusion

This paper deals with the initial-boundary problems for parabolic systems with multiple nonlinear terms of mixed type's i.e, combinations of nonlinearities of power and exponent types. The growths of the solutions come from both the sources and the boundary flux. By using the comparison principle, the blow-up criteria are established under different couplings with nonlinear terms of power or exponent types.

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