Limit Cycle for One Class of Integrable System

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Abstract. We investigate limit cycle for one class of integrable systems. First the problem of number of the Abel integral zero is turned into one of the polynomial zero. Then the least upper bound of number of limit cycle 1 is proved.

Keywords: Limit cycle; zero; inerrable system; linear combination

1. Introduction

We consider a class of inferable systems as follows

$$\begin{aligned} \frac{dx}{dt} &= xy + \varepsilon f(x,y) \\ \frac{dy}{dt} &= \frac{1}{2} - \frac{1}{2}x^2 + 2y^2 + \varepsilon g(x,y) \end{aligned} \tag{1}$$

where $0 < \varepsilon \ll 1$, $f(x,y) = \sum_{i+j=3} a_{ij}x^i y^j, g(x,y) = \sum_{i+j=3} b_{ij}x^i y^j$.

$(a_{ij}, b_{ij} \in R, i, j \geq 0)$.The system (1) has two centre point (1,0) and (-1,0), and has three pairs of infinity point. The parametric equation of the center point (1,0) of closed rail cluster is:

$$\begin{aligned} x_1(t) &= \frac{1-h}{\left[(1-h)^2 + (2h-h^2)\cos t\right]^{\frac{1}{2}}} \\ y_1(t) &= \frac{(2h-h^2)\sin t}{2\left[(1-h)^2 + (2h-h^2)\cos t\right]} \end{aligned} \tag{2}$$

where $h \in \left(0, 1-\frac{\sqrt{2}}{2}\right)$.

The parametric equation of closed rail cluster which surrounds the center point (-1,0) is

$$x_2(t) = -x_1(t), y_2(t) = y_1(t).$$

If $h = 0$, the equation (2) becomes two isolated center singularities (1,0) and (-1,0); if $h = 1 - \frac{\sqrt{2}}{2}$, the equation (2) becomes

$$\begin{aligned} x(t) &= \frac{1}{(1+\cos t)^{\frac{1}{2}}} \\ y(t) &= \frac{\sin t}{2(1+\cos t)} \end{aligned},$$

That is hyperbola $2x^2 - 4y^2 = 1$. If $h \in \left(0, 1-\frac{\sqrt{2}}{2}\right)$, we can get the first integral of the system (1) $H(x,y)$ and integral factor $M(x,y)$

$$H(x,y)=x^{-4}\left(\frac{1}{2}y^2-\frac{1}{4}x^2+\frac{1}{8}\right)=\frac{1-2(1-h)^2}{8(1-h)^4}$$

$$M(x,y)=x^{-5} \tag{3}$$

The author study Hilbert's 16th problem and bifurcations of planar polynomial vector fields in [1]. The authors of [2,3] consider a cubic system with twelve small amplitude limit cycles and Small limit cycles bifurcating from fine focus points in quadratic order Z3-equivariant vector fields. As for the system (1), the mathematics further study on the problem of center and the rail line branch in [4]. Different from the existing research methods, the author puts forward the algebraic method to solve the number of Abel integral's number of zero, with the help of auxiliary calculation of Mathematical, consequently, it makes the study of limit cycles bifurcations from qualitative to quantitative, decreasing the solving difficulty.

2. The Linear Expression of Abel Integral

Definition 1. We consider polynomials systems as follows

$$\frac{dx}{dt}=P(x,y)+\varepsilon f(x,y)$$
$$\frac{dy}{dt}=Q(x,y)+\varepsilon g(x,y) \tag{4}$$

where $0<\varepsilon\le 1$If system (4) has the first integral of the system (1) $H(x,y)$ and integral factor $M(x,y)$

$$P(x,y)=\frac{H_y(x,y)}{M(x,y)}$$

$$Q(x,y)=-\frac{H_x(x,y)}{M(x,y)}$$

system (4) is called integral systems. If system (4) has one center, $I(h)$ is called Abel integral. $I(h)$ as follows

$I(h)=\int_{\Gamma_h}\left[M(x,y)g(x,y)\right]dx-\left[M(x,y)f(x,y)\right]dy$ Γ_h is closed rail of system, namely

$$\Gamma_h\subset\left\{(x,y)\in R^2\,\middle|\,H(x,y)=h,h\in\Sigma\right\}$$

Now, we will show several basic integration forms of the linear combination of Abel integral $I(h)$.

Lemma 1 Let $f(x,y),g(x,y)$ are polynomials of no higher than n times. Denote

$$I_{i,j}(h)=\oint_{\Gamma_h}M(x,y)x^iy^jdx$$

$$J_{i,j}(h)=\oint_{\Gamma_h}M(x,y)x^iy^jdy$$

such that

$$I_{i,j}=-\frac{j}{2(i+2j-4)}\left(-I_{i+2,j-2}+I_{i,j-2}\right)$$

$$J_{i,j}(h)=-\frac{i-5}{j+1}I_{i-1,j+1}$$

Proof: In the equation (2), because of $x_k(t)(k=1,2)$ parity so when j is even number, $I_{i,j}(h)=0$; when j is odd number, by the equation (3) we can know

$$x^{-4}ydy-\left(2x^{-5}y^{2}-\frac{1}{2}x^{-3}+\frac{1}{2}x^{-5}\right)dx=0$$

Both sides are multiplied by $x^{i-4}y^{j}$, according to Γ_h integration we can have

$$\frac{i+2j-4}{j+2}I_{i-4,j+2}-\frac{1}{2}I_{i-2,j}+\frac{1}{2}I_{i-4,j}=0$$

such that

$$I_{i,j}=-\frac{j}{2(i+2j-4)}\left(-I_{i+2,j-2}+I_{i,j-2}\right)$$

By the method of division integral again we can know that

$$J_{i,j}(h)=\oint_{\Gamma_h}M(x,y)x^{i}y^{j}dy=\oint_{\Gamma_h}x^{i-5}y^{j}dy$$

$$=\oint_{\Gamma_h}\frac{x^{i-5}}{j+1}dy^{i+1}=-\frac{i-5}{j+1}I_{i-1,j+1}$$

That is to say, the lemma1 is demonstrated.

Lemma 2. We can show the Abel integral $I(h)$ of the system (1) by following formula

$I(h)=\frac{\pi}{16(h-1)^4}\left[-2b(h-1)^4+a(h-2)^2h^2-2b(h-1)^2\sqrt{2h^2-4h+1}\right]$ Proof By the lemma 1 we can conclude that

$$I_{3,0}=I_{1,2}=0,I_{0,3}=-\frac{3}{4}(I_{0,1}-I_{2,1}),J_{3,0}=2I_{2,1}$$

$$J_{2,1}=\frac{3}{2}I_{1,2}=0,J_{1,2}=-I_{0,1}+I_{2,1},J_{0,3}=\frac{5}{4}I_{-1,4}=0$$

such that

$$\begin{aligned}I(h)&=\oint_{\Gamma_h}M(x,y)g(x,y)dx-M(x,y)f(x,y)dy\\&=b_{30}I_{3,0}+b_{12}I_{1,2}+b_{21}I_{2,1}+b_{03}I_{0,3}-\\&\left(a_{30}J_{3,0}+a_{12}J_{1,2}+a_{21}J_{2,1}+a_{03}J_{0,3}\right)\\&=\frac{1}{4}\left(aI_{0,1}-bI_{2,1}\right)\end{aligned}$$

where $a=4a_{12}-3b_{30},b=4a_{12}+8a_{30}-3b_{03}-4b_{21}$.

Let $\rho=\frac{2h-h^2}{(1-h)^2}$, such that

$$x=\frac{1}{\sqrt{1-\rho\cos t}},y=-\frac{\rho\sin t}{2(1-\rho\cos t)}$$

$$M(x,y)=(1-\rho\cos t)^{\frac{5}{2}}\quad(0\le t\le 2\pi)$$

then

$$\begin{aligned}I(h)&=2\int_0^{\pi}M(x,y)\times\frac{1}{4}\left(axy^2-bx^3y^2\right)dt\\&=\frac{\pi}{16(h-1)^4}\left[-2b(h-1)^4+a(h-2)^2h^2-2b(h-1)^2\sqrt{2h^2-4h+1}\right]\end{aligned}\tag{5}$$

3. The Estimation of the Number of Limit

If $h \in \left(0, 1-\frac{\sqrt{2}}{2}\right)$, we have that

$$\frac{\pi}{16(h-1)^4} > 0$$

By the equation (5) we can know that in order to discuss the number of Abel integral $I(h)$ of zero, we just should consider the function

$F(h) = -2b(h-1)^4 + a(h-2)^2 h^2 - 2b(h-1)^2 \sqrt{2h^2-4h+1}$ If $a=1$, let

$$h = -\sqrt{\frac{1}{2}\left(1+j^2\right)} + 1$$

where $0 < j^2 < 1$,such that

$$F(h) = \frac{1}{4}\left(j^2-1\right)^2 - \frac{b}{2}\left(1+j^2\right)^2 - b\sqrt{j^2} - b\sqrt{j^6}$$

If $j > 0$, we have that

$$F(h) = -\frac{1}{4}(1+j)^2\left[2b\left(1+j^2\right) - (j-1)^2\right]$$

Let $F(h) = 0$, such that

$$j = \frac{-1-2\sqrt{b-b^2}}{2b-1}$$

or

$$j = \frac{-1+2\sqrt{b-b^2}}{2b-1}$$

If $j < 0$, we have that

$$F(h) = -\frac{1}{4}(j-1)^2\left[2b\left(1+j^2\right) - (j+1)^2\right]$$

Let $F(h) = 0$, such that

$$j = \frac{1-2\sqrt{b-b^2}}{2b-1}$$

or

$$j = \frac{1+2\sqrt{b-b^2}}{2b-1}$$

If $0 < j^2 < 1$, let $F(h) = 0$, we have that

$$j = \frac{-1+2\sqrt{b-b^2}}{2b-1}$$

and

$$j = \frac{1-2\sqrt{b-b^2}}{2b-1}$$

Obviously, $j > 0$ and $j < 0$, the solution of $F(h) = 0$ is the opposite number. Therefore, if $a \neq 0$, Abel integral $I(h)$ has one zero at most in definitional domain $h \in \left(0, 1-\frac{\sqrt{2}}{2}\right)$.

If $a = 0$, let $h = -\sqrt{\frac{1}{2}\left(1+j^2\right)} + 1$

where $0 < j^2 < 1$,such that

$$F(h) = -\frac{b}{2}\left(1+j^2\right)\left(1+j^2+2\sqrt{j^2}\right)$$

If $j > 0$, we have that

$$F(h) = -\frac{b}{2}\left(1+j^2\right)\left(1+j^2+2j\right)$$

Let $0 < j^2 < 1$, then we have that $F(h) = 0$ is no solution.

If $j < 0$, we have that

$$F(h) = -\frac{b}{2}\left(1+j^2\right)\left(1+j^2-2j\right)$$

Let $0 < j^2 < 1$, then we have that $F(h) = 0$ is no solution.

So if $a = 0$ Abel integral $I(h)$ is no zero in the domain, and the limit cycle of the system does not exist.

Summing up the above, Abel integral $I(h)$ has one zero at most in definitional domain $h \in \left(0, 1-\frac{\sqrt{2}}{2}\right)$. If $0 < \varepsilon \le 1$, a limit cycle will be bifurcated from system (1) at most. So we can obtain easily the following theorem.

Theorem 1 Set $f(x, y), g(x, y)$ as the cubic homogeneous Polynomial, if the Abel integral $I(h)$ of system (1) is not identically equal to zero, the number of zero of $I(h)$ is at most only one.

So far, it is still difficult to solve the problem of number of zero of the Abel integral, especially to system of inferable non-Hamilton. It is necessary to study by using of updated method and techniques, or we can adopt other methods to tell upper bound of limit cycle numbers for polynomial system rather than by solving Abel integral. Once the problem is solved, it will probably create a new research approach to Hilbert 16th problem of polynomial system.

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