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Three-Dimensional Extended Kantorovich Method Using A Kind of Function Approximation Form with Algebraic Parameters

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Abstract

The paper uses a kind of function approximation form with algebraic parameters which is expressed as $u(x,y,z)=C_1X_1Y_1Z_1+C_2X_2Y_2Z_2$ and designs an iteration scheme in order to achieve three-dimensional extended Kantorovich method which has the iteration convergence. However, the numerical example of three-dimensional Poisson equation demonstrates that the iteration process doesn't converge and the method doesn't have the iteration convergence.

Keywords

Function approximation; algebraic parameters; extended Kantorovich method; Poisson equation.

1. Introduction

In 1958, the Kantorovich method[1] was presented as a kind of semi-analytical method which was discretized in one direction and was analyzed in another direction. Furthermore in 1968, Kerr[2,3] presented the extended Kantorovich method (EKM) which was analyzed in both of two directions to solve the elastic torsion problem of the bar with rectangular section and the bending problem of the thin plate with four clamped edges.

In the recent years, reference [4] employed EKM for static analysis of functionally graded rectangular plates, reference [5] employed EKM for vibration analysis of laminated plates and reference [6] employed EKM for the characterization of the static behavior of electrically actuated micro-plates. Reference [8] used the function approximation of tensor product form and achieved three-dimensional(3D) EKM of tensor product form.

Reference [7] demonstrated that the iteration process didn't converge if the simple function approximation form similar with 2D EKM was employed. In order to realize the convergence of the iteration process, the paper presents a kind of function approximation form with algebraic parameters in which there are several univariate functions in 3D and additionally several algebraic parameters. The goal is to achieve 3D EKM of function approximation form with algebraic parameters.

2. Function approximation form

The paper uses the following function approximation form:

$$u(x,y,z)=C_1X_1(x)Y_1(y)Z_1(z)+C_2X_2(x)Y_2(y)Z_2(z) \qquad (1)$$

The function approximation form in formula (1) has the feature of the symmetry in 3D and the number of univariate functions in 3D is equal. Therefore, in an iterative circle there are three iterative steps of solving ODEs and one iterative step of solving AEs.

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3. Algorithm implementation

3.1. Integral and differential equations

Dirichlet problem for Poisson Equation on the 3D cubic domain is taken as the example. The algorithm implementation of 3D EKM of function approximation form with algebraic parameters is demonstrated here. The energy functional of 3D Poisson Equation is as follows:

$$\begin{cases} \pi(u)=\dfrac{1}{2}\iiint_{\Omega}\left[\left(\dfrac{\partial u}{\partial x}\right)^2+\left(\dfrac{\partial u}{\partial y}\right)^2+\left(\dfrac{\partial u}{\partial z}\right)^2\right]dxdydz-\iiint_{\Omega} f\ u\ dxdydz \\ s.t. \quad u=0 \quad on \quad x=\pm a,\ y=\pm b,\ z=\pm c \end{cases} \tag{2}$$

where Ω is the regular domain of $[-a,a]\times[-b,b]\times[-c,c]$.

When the function approximation in formula (1) is substituted into the energy functional in formula (2) and the variation operation of the energy functional is performed, a set of coupled integral and differential equations is derived as follows:

$$[A_1]\{X''\}-[B_1]\{X\}+\{F_1\}=\{0\} \tag{3a}$$

$$[A_2]\{Y''\}-[B_2]\{Y\}+\{F_2\}=\{0\} \tag{3b}$$

$$[A_3]\{Z''\}-[B_3]\{Z\}+\{F_3\}=\{0\} \tag{3c}$$

Boundary conditions are as follows:

$$\{X(-a)\}=\{0\},\quad \{X(a)\}=\{0\} \tag{4a}$$

$$\{Y(-b)\}=\{0\},\quad \{Y(b)\}=\{0\} \tag{4b}$$

$$\{Z(-c)\}=\{0\},\quad \{Z(c)\}=\{0\} \tag{4c}$$

The coefficient matrixes of ODEs in formula (3a) are as follows:

$$[A_1]=\int_{-b}^{b}\int_{-c}^{c}\begin{bmatrix} G_1^{\,2} & G_1G_2 \\ G_1G_2 & G_2^{\,2} \end{bmatrix}dydz$$

$$[B_1]=\int_{-b}^{b}\int_{-c}^{c}\begin{bmatrix} \left(\dfrac{\partial G_1}{\partial y}\right)^2+\left(\dfrac{\partial G_1}{\partial z}\right)^2 & \dfrac{\partial G_1}{\partial y}\cdot\dfrac{\partial G_2}{\partial y}+\dfrac{\partial G_1}{\partial z}\cdot\dfrac{\partial G_2}{\partial z} \\ \dfrac{\partial G_1}{\partial y}\cdot\dfrac{\partial G_2}{\partial y}+\dfrac{\partial G_1}{\partial z}\cdot\dfrac{\partial G_2}{\partial z} & \left(\dfrac{\partial G_2}{\partial y}\right)^2+\left(\dfrac{\partial G_2}{\partial z}\right)^2 \end{bmatrix}dydz \tag{5}$$

$$\{F_1\}=\int_{-b}^{b}\int_{-c}^{c} f\begin{Bmatrix} G_1 \\ G_2 \end{Bmatrix}dydz$$

Where $G_1=Y_1(y)\cdot Z_1(z)$, $G_2=Y_2(y)\cdot Z_2(z)$.

Note that the coefficient matrixes of ODEs in formula (3b) can be obtained through replacing $Y(y), Z(z)$ in formula (5) with $Z(z), X(x)$. The coefficient matrixes of ODEs in formula (3c) can be obtained through replacing $Y(y), Z(z)$ in formula (5) with $X(x), Y(y)$.

$[A_1],[A_2],[A_3],[B_1],[B_2]$ and $[B_3]$ are all the matrixes of order 2 and $\{F_1\}$, $\{F_2\}$ and $\{F_3\}$ are all the vectors of order 2 in formula (3a), (3b) and (3c).

On the basis of three ODEs of formula (3a), (3b) and (3c), the paper adds the AEs about algebraic parameters as follows.

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The function approximation form in formula (1) is changed equivalently into the following function approximation form:

$$u(x,y,z)=(C_{x1}X_1+C_{x2}X_2)(C_{y1}Y_1+C_{y2}Y_2)(C_{z1}Z_1+C_{z2}Z_2)+(C_{x3}X_1+C_{x4}X_2)(C_{y3}Y_1+C_{y4}Y_2)(C_{z3}Z_1+C_{z4}Z_2) \quad (6)$$

When the function approximation in formula (6) is substituted into the energy functional in formula (2) and the variation operation of the energy functional is performed, the nonlinear AEs about $\{C_{x1},C_{x2},C_{x3},C_{x4}\}$, $\{C_{y1},C_{y2},C_{y3},C_{y4}\}$ and $\{C_{z1},C_{z2},C_{z3},C_{z4}\}$ are derived.

3.2. Iteration Step

Because the functions in three dimensions are symmetrical in formula (1) of the function approximation form, the computation will arrange the iteration steps in the order of dimensions and in every iteration circle, an iteration step is added to solve algebraic equations for $\{C\}$. As shown in Figure 1, an iteration circle is made up of four steps of $(a)\rightarrow(b)\rightarrow(c)\rightarrow(d)$: (In general, step (a) is that if $\{X\}$, $\{Y\}$ are known, solve for $\{Z\}$.)

(a) If $\{X\}$, $\{Y\}$ are fixed, the integral and differential equations (IDEs) of formula (3),(4) are degenerated to ordinary differential equations (ODEs) of formula (3c),(4c). Solve the ODEs in order to obtain $\{Z\}$;

(b) If $\{Y\}$, $\{Z\}$ are fixed, IDEs of formula (3),(4) are degenerated to ODEs of formula (3a),(4a). Solve the ODEs in order to obtain $\{X\}$;

(c) If $\{Z\}$, $\{X\}$ are fixed, IDEs of formula (3),(4) are degenerated to ODEs of formula (3b),(4b). Solve the ODEs in order to obtain $\{Y\}$;

(d) If $\{X\}$, $\{Y\}$, $\{Z\}$ are fixed, solve the AEs in order to obtain algebraic parameters $\{C\}$.

On the basis of the iteration step (a),(b),(c), the paper adds the iteration step (d) to solve the nonlinear AEs about $\{C_{x1},C_{x2},C_{x3},C_{x4}\}$, $\{C_{y1},C_{y2},C_{y3},C_{y4}\}$ and $\{C_{z1},C_{z2},C_{z3},C_{z4}\}$.

Through the process of $(a)\rightarrow(b)\rightarrow(c)\rightarrow(d)$, an iteration cycle is finished. Check whether the error of numerical solution is less than the tolerance. If yes, then finish the iteration. Otherwise return to step (a) to begin the next iteration cycle.

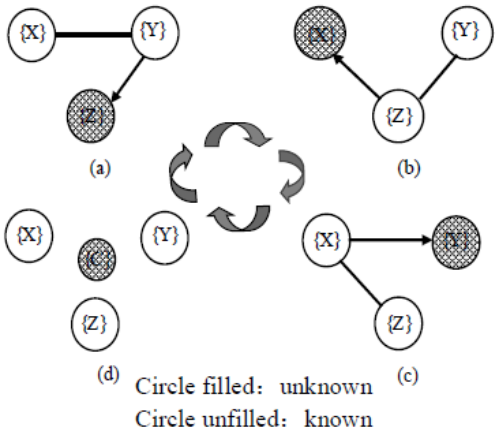

Fig 1. An iteration process including four steps of $(a)\rightarrow(b)\rightarrow(c)\rightarrow(d)$

4. Numerical example

Example: Dirichlet problem for three-dimensional Poisson equation (Fig 2)

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Partial differential equation (PDE) follows as:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -f$$

Boundary conditions (BCs) follow as:

$u = 0 \quad on\ x = \pm a,\ y = \pm b,\ z = \pm c$

Let $a = b = c = 1$, $f = 2$

Let the initial function be

$$Y_i(y) = \cos\left(\frac{(2i-1)y\pi}{2b}\right), \quad X_i(x) = \cos\left(\frac{(2i-1)x\pi}{2a}\right), \quad i = 1,2,\cdots,n$$

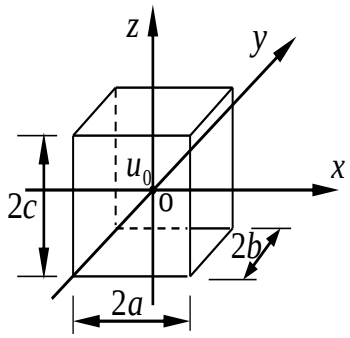

Fig2 three-dimensional Poisson equation

Numerical results are shown in the following figure. Note that an iteration circle is made up of four iteration steps.

Table 1 shows the displacement of centre point and the energy functional in the first 7 iteration circles to observe the iteration convergence. In Fig 3 and Fig 4, the iteration step (d) which is added to solve for AEs can decrease significantly the energy functional in the first 2 iteration circles. However, in the next iteration circles the displacement of centre point and the energy functional change in the uniform speed and don't have the tendency of iteration convergence. This phenomenon of the non-convergence is the same as the phenomenon of the non-convergence in reference [7].

Table 1 The displacement of centre point and energy functional in the first 7 iteration circles

No of iteration circles	Displacement of centre point u_0	Displacement error (%)	Energy functional	Energy error (%)
1	0.44512655	-0.486	-1.28992532	-0.05645
2	0.44698830	-0.07	-1.29062704	-0.00208
3	0.44705901	-0.054	-1.29063488	-0.00147
4	0.44713108	-0.038	-1.29064087	-0.00101
5	0.44719568	-0.024	-1.29064590	-0.00062
6	0.44725226	-0.011	-1.29065017	-0.00029
7	0.44730230	0	-1.29065385	0

Note: assume the solution of the 7th iteration circle as the exact solution for calculating the errors.

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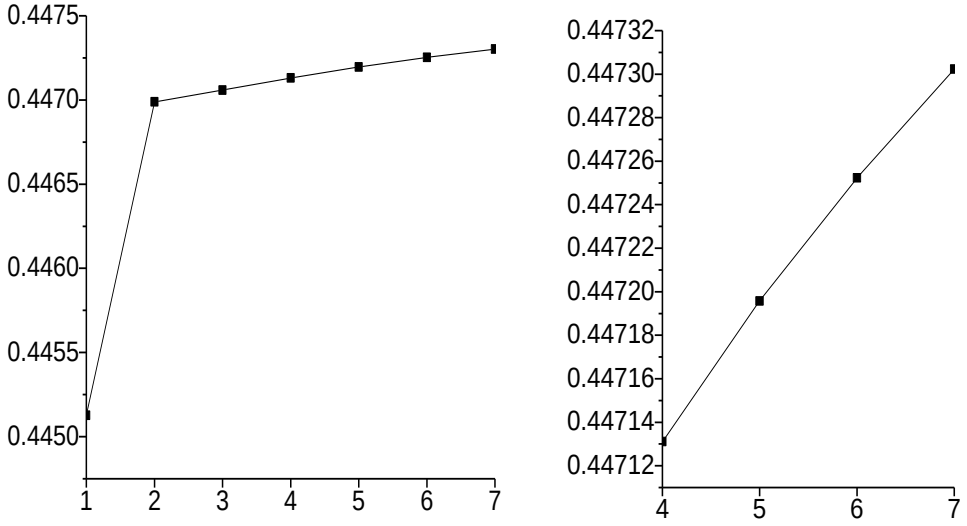

Fig 3 The displacement of centre point in the first 7 iteration circles
(the right figure: the drawing of partial enlargement for the last 4 iteration circles)

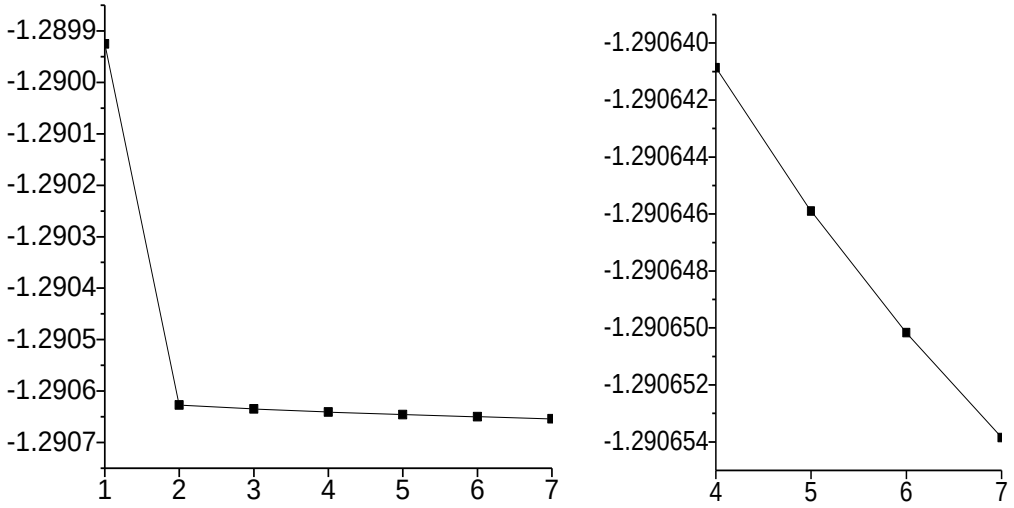

Fig 4 The energy functional in the first 7 iteration circles
(The right figure: the drawing of partial enlargement for the last 4 iteration circles)

5. Conclusion

About 3D EKM of function approximation form with algebraic parameters, there are two following conclusions:

On the basis of the iteration step (a),(b),(c), the paper adds the iteration step (d) to solve the nonlinear AEs. Numerical results demonstrated that the iteration step (d) which is added can decrease significantly the energy functional in the first 2 iteration circles. However, in the next iteration circles the displacement of centre point and the energy functional change in the uniform speed and don't have the tendency of iteration convergence.

Reference [8] demonstrated that three-dimensional(3D) EKM of tensor product form can realize the iteration convergence. Through comparing the non-convergence in the paper with the convergence in reference [8], it is observed that the function approximation of tensor product form is an important influencing factor for the iteration convergence. When the appropriate function approximation form is used, the EKM is an algorithm of high accuracy and high efficiency to solve for 3D problem.

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