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Parameter Range Estimation by Grid Method for Endomorphism Conditions in Bimodal Maps

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Abstract

It is well known that symbolic dynamics is based on the iterated maps of endomorphism on a given interval. Although coarse granulation leads information lost, however for a given periodic super stable kneading sequence (SSKS) , the corresponding parameters of iterated endomorphism map may be calculated by the word-lifting algorithm or technique, it is to solve a system of nonlinear equations with m variables for m-modal maps. The famous tough issue is initial sensitivity. In the paper, for m takes 2, the bimodal map is considered for the simplest case. The endomorphism conditions are firstly used to obtained the parameter range estimation by grid method, the final goal is to obtain a proper parameter range for selection of initial value, it will alleviate the hardness of initial problem. On the other hand, the boundaries of SSKS in bimodal maps are drawn and four corresponding equations are obtained by numerical method.

Keywords

Symbolic dynamics; Grid method; Endomorphism conditions; Parameter range estimation.

1. Introduction

Symbolic dynamics can be constructed with an endomorphism iterative map by a coarse granulation process. Symbolization of a numerical orbit looks like information lose, Kaplan [1] had proposed an algorithm to solve the inverse problem firstly, it was developed by B.-L. Hao in symbolic dynamics of unimodal maps later and named it the word-lifting algorithm [2]. With the development of the 1D symbolic dynamics, the difficult of lacking explicit express in inverse function of polynomials to the 5th power or more leads to no way to research the symbolic dynamics in 1D m-modal maps if $m \geq 4$. We had proposed an effective numerical algorithm and solved this problem [3], in fact, the new word-lifting algorithm is quick and effective because of using Newton's iterative method of second-order convergence, the key issue is a proper initial point should be given in the parameter space, else the iterative process may lead to a divergence. The traditional method to find a proper initial point is empirical and comes from the geometric meaning of parameters. For example, in an iterative system with bimodal maps, c and d represents the horizontal coordinate of critical point C and D, it satisfies a non-equality $-1 < c < d < 1$. In fact, we observe a phenomenon during some newton iteration process which can not attain the fixed point, the map value often beyond the range of endomorphism interval [-1,1], this kind of divergence accounts for a large proportion, of course the primary reason is the selection of an unfit initial point. So, the distribution diagram of initial points should be researched and drawn, it will boost a convergence probability of the newton iterative method by such an empirical initial point. In the paper, some endomorphism condition for bimodal maps are proposed, two parameters c and d form a rectangle region in a plane by the geometric meaning, the region is divided into a dense grid, for every cross point, the corresponding map is calculated, if the endomorphism condition is satisfied, the cross point is marked as green color else as red color on the plane graph. If the partition is fine enough, a graph with red and green color is finished, it is called endomorphism region graph (ERG). The result is that if the initial point you selected falls into the red region, the newton iteration must go into the orbit of divergence, while if falls into the green region, the newton iteration will converge or diverge, apparently the selection in green region will enlarge the chance of convergence during the newton iteration process.

On the other hand, we know the SSKS is called 'joint' in symbolic space and played an important role in star product [4]. All the parameters of SSKSs in symbolic dynamics of 1D bimodal maps are calculated by the new word-lifting algorithm in the parameter space, in fact it is a parameter plane here. The boundaries are approximate four lines, their equations are presented in the paper by fitting method. The ERG and 'joint' graph are drawn in a same plane, it is a interesting thing.

The paper is organized as follows. In sec.2, iteration modal map with endomorphism interval and symbolic dynamics of bimodal maps is introduced, ERG is obtained by the endomorphism conditions and the grid method; in sec.3, the parameters of two type of SSKS with period 1-12 are calculated by word-lifting algorithm, while the boundaries of 'joint' in the parameter plane are fitted into four linear equations; in sec.4, conclusion is given.

2. Producing ERG With Endomorphism Conditions and Grid Method

2.1. The Iteration Modal and Symbolic Dynamics of Bimodal Maps

Consider the general bimodal maps $f(x)=k\int(x-c)(x-d)dx+i$, on the interval [-1,1] of endomorphism, by two boundary conditions $f(-1)=-1$ and $f(1)=1$, $k=\frac{3}{1+3cd}$, $i=\frac{3(c+d)}{2(1+3cd)}$, parameters k and i are eliminated. In fact, the iterated map of bimodal is written as

$$x_{n+1}=f(x_n,c,d) \tag{1}$$

Here, c and d are horizontal coordinates of two critical points C and D, while L,M and R are three monotonous limbs, by the MSS order [5], $L\prec C\prec M\prec D\prec R$ holds. For an initial point x_0, by iterative map (1), a numerical orbit is obtained as x_0, x_1, ... , x_n, it can be converted into a symbolic sequence $S_0S_1\cdots S_n\cdots$, by the following rule (2), the coarse granulation process is quintessence of the symbolic dynamics [6].

$$S_k=\begin{cases} L, & if\ x_k<c, \\ C, & if\ x_k=c, \\ M, & if\ c<x_k<d,\quad k\in Z^+ \\ D, & if\ x_k=d, \\ R, & if\ x_k>d. \end{cases} \tag{2}$$

If the sequence is periodic, for example $\left(XDYC\right)^{\infty}$ and $\left(\left(XD\right)^{\infty},\left(YC\right)^{\infty}\right)$ are two type of SSKS and they are called 'joint' in the symbolic space, while they are simply normalized as $XDYC$ and $\left(XD,YC\right)$, the former is single cycle and the latter is with two cycles, where X,Y are sequences composed of $\{L,M,R\}$, these sequences are periodic and passed through the two critical points C and D respectively. Other notations and concepts such as sorting order rule of sequence, shift operator and the word-lifting algorithm are omitted here [7].

2.2. Endomorphism Conditions and Grid Method

An endomorphism map is defined as f: $G \to G$, 1D bimodal maps should let G is I, I takes [-1, 1] without loss of generality,

$$f : I \to I \tag{3}$$

1D symbolic dynamics requires the kernel map should be endomorphism, for $x_0 \in I$, by (1) and (3), $x_k \in I, k = 1, 2, \cdots$. It will hold the structure of algebra and topology, after symbolization (2), symbolic dynamics may describe real dynamic system, otherwise the calculation of topological entropy, orbit analysis would lose the premise. For an initial point (c_0, d_0), the following inequalities and two boundary conditions forms the endomorphism conditions:

$$\begin{cases} -1 \le f(d) < f(c) \le 1 \\ -1 < c < d < 1 \\ f(-1) = -1, \ f(1) = 1 \end{cases} \tag{4}$$

Two boundary conditions imply that the three monotonic limbs L, M and R is (+-+) type respectively, see Fig.1. Another case (-+-) is trivial, we do not discuss in this paper.

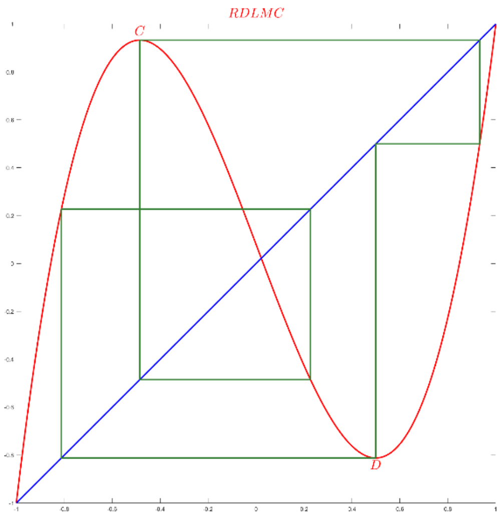

Fig. 1 SSKS RDLMC for +-+case

2.3. Producing ERG By Endomorphism Conditions and Grid Method

First, a Cartesian coordinate system is established, c is denoted as x axis and d as y axis respectively. $[-1,1]\times[-1,1]$ square is meshed into n^2 grid cross points and $(n-1)^2$ grids. Second, for every cross point, it is checked by condition (4), if it satisfies and mark the point as 1, else marked as 0. After visited all the n^2 grid cross points, we have obtained all the marked variables Z. Finally, the points marked as 1 are drawn as green color, others as red color. The ERG has been finished. The goal of ERG is useful for proper selection of initial point before the newton-iteration method starts. Points on the red area would lead to a divergence at once, the green area should be considered, although it can ensure the convergence. It will enhance the probability of convergence. Here we presented the Matlab codes for the deeper understanding the implement of ERG.

```
function y1 = if_in(c,d)
  y1 = 1;
  yc = f1(c,c,d);
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    yd = f1(d,c,d);
    if ((c>=d) || (yc>1) || (yd<-1) || (yc<=yd))
        y1=0;
    end
end
function yy=f1(xx,c,d)
    k=1/(1/3+c*d);
    i=3/2*(c+d)/(1+3*c*d);
    yy=k/3*xx.^3-k/2*(c+d)*xx.^2+k*c*d*xx+i;
end
n = 120;
x = linspace(-1,1,n);
y = linspace(-1,1,n);
[X,Y] = meshgrid(x,y);
Z = zeros(size(X));
for row = 1:n
    for col=1:n
        Z(row,col) = if_in(X(row,col),Y(row,col));
    end
end
x1 = reshape(X,1,n*n);
y1 = reshape(Y,1,n*n);
z1 = reshape(Z,1,n*n);
t1 = find(z1==1);
t2 = find(z1==0);
scatter(x1(t1),y1(t1),36,[0 1 0],'filled')
hold on
scatter(x1(t2),y1(t2),36,[1 0 0],'filled')
```

The codes above may produce the ERG (see Fig.2).

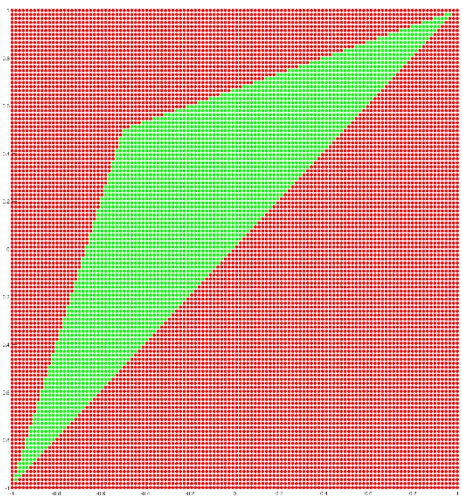

Fig. 2 ERG for 1D bimodal maps

If it takes n = 1000, the more detailed and high definition of ERG is obtained. It is obvious that the green region looks like a rectangle. It may be fitted by the Matlab function convhull(x1(t1),y1(t1)) which is contextual association with Matlab codes above. Three boundary equations are as follows:

$$\begin{cases} d = c, c \in (-1,1) \\ d = 3c + 2, c \in (-1,-0.5) \\ d = \frac{1}{3}c + \frac{2}{3}, c \in (-0.5,1) \end{cases} \tag{5}$$

The green domain is an open set of the points apparently, the points on the three boundaries do not satisfy endomorphism conditions (4). In addition, points away from the border may be more easier to fall into the endomorphism interval on the subsequent iterations. We know that the SSKS is the 'joint' in the symbolic space and played an important role on the study of symbolic dynamics, in addition, other 'skeleton' or 'muscle' are developed with 'joint' [4], if the parameters of the SSKS have been calculated and drawn in a same plane with ERG, it will enhance the knowledge of selection of initial point.

3. Drawing the 'joints' on the parameter plane with ERG

3.1. Calculation of Parameters for Two Type SSKS $XDYC$ and (XD,YC)

For a given periodic sequence $W, n = |W|$ is the period of W, all the permutation of sequences can be produced as set κ_0^n. By the admissibility conditions,

$$\begin{cases} \bar{L}(W) \prec \bar{C}(W), \bar{M}(W) \prec \bar{C}(W), \\ \bar{D}(W) \prec \bar{M}(W), \bar{D}(W) \prec \bar{R}(W). \end{cases} \tag{6}$$

Where $\bar{S}(W)$ stands for all subsequences of symbol S and $S \in \{L, C, M, D, R\}$, $\prec$ is the MSS order [5]. It is worthing of noting that $\bar{C}(W) = XD$ and $\bar{D}(W) = YC$ for $XDYC$ type, on the other hand, $\bar{C}(W) = YC$ and $\bar{D}(W) = XD$ for (XD,YC) type. For $\forall W \in \kappa_0^n$, if it meets the requirement of (6), it is called admissible SSKS of n-period, all the admissible SSKS form a set κ_1^n. Here, we presented a table for the count of admissible SSKS in κ_1^n. Every element in κ_1^n corresponds a real numerical orbit.

Table 1 The count of elements in κ_1^n for n takes value 2-15

Period n	$XDYC$	(XD,YC)	Total Counts
2	1	0	1
3	2	0	2
4	5	1	6
5	12	4	16
6	30	14	44
7	78	44	122
8	205	129	334
9	546	372	918
10	1476	1064	2540
11	4026	3020	7046
12	11070	8554	19624
13	30660	24256	54916
14	85410	68862	154272
15	239144	195868	435012

3.2. Calculation the Parameters for W in κ_1^n

Every SSKS in κ_1^n corresponds to a system of nonlinear equations which can be solved by newton-iteration method with the new word-lifting algorithm [], the calculation process is omitted here. 30653 SSKSs for period 1-12 in κ_1^n is involved in the calculation, only four SSKS runs divergence, the reason still come from the initial value sensitivity. In fact, for period 13-15 in κ_1^n more and more SSKSs would be in a difficult position to calculate for the same reason, in spite of the low proportion. However, for m-modal map $m \geq 3$ the initial value sensitivity becomes more intractable. This the goal of the paper, to acquire more knowledge for the parameter distribution. 30649 points are drawn with blue color while the EGR boundaries with green lines are drawn with the same coordinate plane. (seen Fig.1 left)

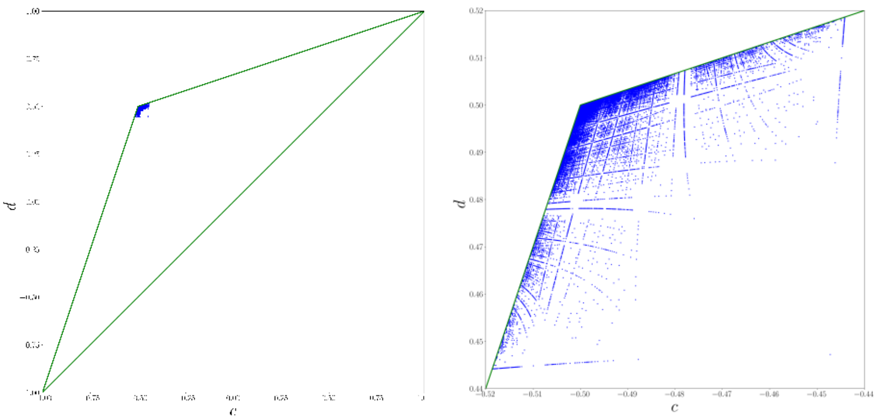

Fig. 3 ERG and 'joints' in a same plane(left); Enlarging the graph for domain $[-0.52,-0.44]\times[0.44,0.52]$(right)

The left graph in Fig.3 shows that the 'joint' area is 0.0064 while the area of the ERG is 1, the area of parameter region only occupies the 0.64 percent of the ERG's. It seems that the EGR has little effect, however, the two green boundary lines turned out to be the boundary of the parameter of 'joints' in Fig.3 apparently. the result shows the simple endomorphism conditions (4) can determine the part of boundaries of the 'joint', the initial points can be selected in the small bule region which is a rhombus, boundary equations are easily fitted by Matlab code, four vertex coordinates of the rhombus boundary are calculated as following

$$\begin{cases}(-0.4627654521266084, 0.4627654521266084),\\(-0.4441474047698563, 0.5186175314666635),\\(-0.4999994841099075, 0.4999994841099075),\\(-0.5186175314666636, 0.4441474047698559).\end{cases} \tag{7}$$

The four vertex coordinates lined to four edges of a rhombus which form a parameter plane of all 'joints' in κ_1^n, the left corner is dense and right corner is sparse. Zoom in on any subregion, the fancy fractal structures are contained within it.

4. Conclusion

The simple endomorphism conditions may produce the boundary of dense region for 'joints' which named SSKS and constructed the kneading space. It will provide a more exact selection of the initial point during the newton-iteration method by the new word-lifting algorithm. The method of producing ERG is very useful for the word-lifting algorithm for 1D m-modal maps in spite that the 99 percent region is useless, the parameter region is resized smaller, it is easier to find proper initial values than before. In fact, if $m \geq 3$ the difficult of parameter calculation will increase violently, there are three critical points, the SSKS have more type and more complicated cases. It is worthy of searching the boundary of parameter space for trimodal maps and word-lifting should be developed deeper and deeper, it will drive the progress of symbolic dynamics.

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