Preliminary test almost unbiased ridge estimator based on W test-statistics

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Abstract. In this paper, we proposed the preliminary test almost unbiased ridge estimator based on W test-statistics, when it is suspected that the regression parameter may be restricted to a space in a restricted linear model with measurement error. The property of the new estimator is also discussed.

Keywords: Preliminary test estimator, Mean squared error, almost unbiased ridge estimator, Measurement error.

1. Introduction

Let us study the linear regression model with measurement error

$$Y_t = \beta_0 + x_t'\beta + e_t, X_t = x_t + u_t, t = 1,2,...,n \tag{1}$$

Where β_0 shows the intercept term, β shows the unknown parameters, $x_t = \left(x_{1t},...,x_{pt}\right)'$, $x_t = \left(x_{1t},...,x_{pt}\right)'$ and measurement error $u_t = \left(u_{1t},...,u_{pt}\right)'$ consist with $X_t = \left(X_{1t},...,X_{pt}\right)'$, u_{it} is the measurement error, e_t stands for the error. Suppose that

$$\left(x_t', e_t, u_t\right) \sim N_{2p+1}\left(\left(u_x', 0, 0'\right)', Blockdiag\left(\Sigma_{xx}, \sigma_{ee}, \Sigma_{uu}\right)\right) \tag{2}$$

For $u_x = \left(u_{x1},...,u_{xp}\right)'$ σ_{ee} shows the variance of e_t Σ_{xx} and Σ_{uu} present the variance matrix of x_t and u_t . Then $\left(Y_t, X_t'\right)'$ follows a normal distribution with mean vector $\left(\beta_0 + \beta' u_x, u_x'\right)'$ and covariance matrix

$$\begin{pmatrix} \sigma_{ee} + \beta'\Sigma_{xx}\beta & \beta'\Sigma_{xx} \\ \Sigma_{xx}\beta & \Sigma_{xx} + \Sigma_{uu} \end{pmatrix} \tag{3}$$

Based on this we get

$$E\left(Y_t \middle| X_t\right) = \gamma_0 + \gamma' X_t \tag{4}$$

For $\gamma_0 = \beta_0 + \beta'\left(I_p - K_{xx}'\right)u_x$ $\gamma = K_{xx}\beta$ $K_{xx} = \Sigma_{xx}^{-1}\Sigma_{xx} = \left(\Sigma_{xx} + \Sigma_{uu}\right)^{-1}\Sigma_{xx}$.

Suppose that the unknown parameter statisfy the following restrictions:

$$H\beta = h \tag{5}$$

Where H denotes a $q \times p$ matrix is a vector of $q \times 1$.

For the linear model with no measurement error, when the statistician suspect the linear restrictions, many authors have studied the preliminary test estimator which is based on Wald(W),Likelihood Ration(LR) and Varangian Multiplier (LM) test-statistic ,such as Yang and Xu [1], Chang and Yang [2-3] et al. For the linear model with measurement error, when the statistician suspect the linear restrictions, we consider the following test. Null hypothesis: $H_0: H\beta = h$, alternative hypothesis $H_1: H\beta \neq h$. Saleh and Shalabh [4] discuss the preliminary test ridge estimator best on W teste-statistic, and the also discussed the properties of the new estimator. In this paper we propose an almost unbiased ridge estimator based on W statistics and almost unbiased ridge estimator, we also discuss the properties of the new estimator.

2. The new estimator

For model (1), one problem is to estimate β. When $\sum_{uu}$ is known: Glaser [5] discuss the estimator of $\gamma_0, \gamma, \sigma_{zz}$:

$$\tilde{\gamma}_{0n} = \bar{Y} - \tilde{\gamma}_n \tilde{X} \ \ \tilde{\gamma}_n = S_{XX}^{-1} S_{XY} \text{ And } \tilde{\sigma}_{zz} = \frac{1}{n}\left(Y - \tilde{\gamma}_{0n} 1_n - \tilde{\gamma}_n' X\right)' \left(Y - \tilde{\gamma}_{0n} 1_n - \tilde{\gamma}_n' X\right) \tag{6}$$

$$\tilde{\sigma}_{ee} = \tilde{\sigma}_{zz} - \tilde{\gamma}_n' K_{xx}^{-1} \textstyle\sum_{uu} \tilde{\gamma}_n \geq 0 \tag{7}$$

Where

$$S = \begin{pmatrix} S_{YY} & S_{YX} \\ S_{XY} & S_{XX} \end{pmatrix}$$

$$S_{YY} = (Y - \bar{Y} 1_p)'(Y - \bar{Y} 1_p), Y = (Y_1, ..., Y_n)', 1_n = (1, ..., 1)'$$

$$S_{XX} = \left((S_{X_i X_i})\right), S_{X_i X_i} = (x_i - \bar{X}_i 1_n)'(x_i - \bar{X}_i 1_n)$$

$$S_{X_i Y} = \left(X_i - \bar{X}_i 1_n\right)' \left(Y_i - \bar{Y}_i 1_n\right), S_{XY} = \left(S_{X_1 Y}, ..., S_{X_p Y}\right)'$$

$$\bar{X}_i = \frac{1}{n} \sum_{i=1}^{n} X_{it}, \bar{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_t$$

When $\sum_{uu}$ and $K_{xx} = \sum_{xx}^{-1} \sum_{xx} = \left(\sum_{xx} + \sum_{uu}\right)^{-1} \sum_{xx}$ are unknown, we use the following estimator to estimate K_{xx}:

$$\hat{K}_{xx} = S_{XX}^{-1} \left(S_{XX} - n \textstyle\sum_{uu}\right) \tag{8}$$

Where $\frac{1}{n} S_{XX}$ presents the maximum likelihood estimator of $\sum_{xx} + \sum_{uu}$.

In this case the estimators of $\beta_0, \beta_1, \sigma_{ee}$ are defined as follows

$$\tilde{\beta}_{0n} = \tilde{\gamma}_{0n} - \tilde{\beta}_n' \left(I_p - \hat{K}_{xx}'\right) \bar{x}, \ \tilde{\beta}_n = \hat{K}_{xx}^{-1} \tilde{\gamma}_n, \ \tilde{\sigma}_{ee} = \tilde{\sigma}_{zz} - \tilde{\beta}_n' \textstyle\sum_{uu} \hat{K}_{xx} \tilde{\beta}_n \tag{9}$$

Where

$$\tilde{\beta}_n = \left(S_{XX} - n \textstyle\sum_{uu}\right)^{-1} S_{XY}$$

By Fuller [6], we have the variance of $\tilde{\beta}_n$ is $\sigma_{zz} C$ where $C = K_{xx}' \sum_{XX} K_{xx} = \sum_{xx}' \sum_{XX} \sum_{xx}$ Then an estimator of C is;

$$C_n = \hat{K}_{xx}' \textstyle\sum_{XX} \hat{K}_{xx}$$

In order to deal with multicollinearity, Saleh and Shalabh [4] proposed the ridge estimator to improve $\tilde{\beta}_n$

$$\tilde{\beta}_n(k) = \left(I_p + k\left(\hat{K}_{xx}' \textstyle\sum_{XX} \hat{K}_{xx}\right)^{-1}\right)^{-1} \tilde{\beta}_n \tag{10}$$

In this paper we use the almost unbiased method and propose an almost unbiased ridge estimator which is defined as follows:

$$\tilde{\beta}_{nAURE}(k) = \left(I_P - k^2 \left(I_p + k\hat{K}_{xx}' \textstyle\sum_{XX} \hat{K}_{xx}\right)^{-2}\right) \tilde{\beta}_n$$

Define $A_n(k)=\left(I_P-k^2\left(kI_p+\hat{K}'_{xx}\sum_{XX}\hat{K}_{xx}\right)^{-2}\right)=\left(I_P-k^2\left(kI_p+C_n\right)^{-2}\right)$, then we can write $\tilde{\beta}_{nAURE}(k)$ as follows:

$$\tilde{\beta}_{nAURE}(k)=A_n(k)\tilde{\beta}_n \quad (11)$$

Consider model (1) and linear restrictions (5), we get the restricted estimator of β

$$\hat{\beta}_n=\tilde{\beta}_n-C_n^{-1}H'\left(HC_n^{-1}H'\right)^{-1}\left(H\tilde{\beta}_n-h\right) \quad (12)$$

When we suspect the linear restrictions, we consider the following W test-statistics, which is defined as

$$L_n=n\left(H\tilde{\beta}_n-h\right)'\left(HC_n^{-1}H'\right)^{-1}\left(H\tilde{\beta}_n-h\right) \quad (13)$$

When null hypothesis $H_0:H\beta=h$ is right $L_n\xrightarrow{D}\chi_q^2$.

Saleh and Shalabh [4] based on W test-statistics and propose the following estimator:

$$\hat{\beta}_n^{PT}=\tilde{\beta}_n-\left(\tilde{\beta}_n-\hat{\beta}_n\right)I\left(L_n<\chi_q^2(\alpha)\right) \quad (14)$$

In this paper we propose a preliminary test almost unbiased ridge estimator based on W test-statistics:

$$\hat{\beta}_{nAURE}^{PT}(k)=A_n(k)\hat{\beta}_n^{PT} \quad (15)$$

In next section, we will discuss the properties of the new estimator.

3. The properties of the new estimator

In this section we will discuss the comparison of preliminary test almost unbiased ridge estimator and preliminary test estimator under the mean squared error criterion.

By (15), we have:

$$E\left(\hat{\beta}_{nAURE}^{PT}(k)\right)=A(k)E\left(\hat{\beta}_n^{PT}\right)=A(k)\beta-A(k)\eta H_{q+2}\left(\chi_q^2(\alpha);\Delta^2\right) \quad (16)$$

Where $A(k)=\left(I_P-k^2\left(kI_p+C\right)^{-2}\right)$, $H_{q+2}\left(\chi_q^2(\alpha);\Delta^2\right)$ denote q degree, non-centrality parameter Δ^2 non-central chi square distribution function, and $\eta=C^{-1}H'\left(HC^{-1}H'\right)^{-1}\left(H\beta-h\right)$.

$$Bias\left(\hat{\beta}_{nAURE}^{PT}(k)\right)=-k^2C^{-2}(k)\beta-\left(I-k^2C^{-2}(k)\right)\eta H_{q+2}\left(\chi_q^2(\alpha);\Delta^2\right)$$

$$\begin{aligned}Risk\left(\hat{\beta}_{nAURE}^{PT}(k)\right)&=\sigma_{zz}tr\left(C^{-1}A^2(k)\right)-\sigma_{zz}tr\left(RA^2(k)\right)H_{q+2}\left(\chi_q^2(\alpha);\Delta^2\right)\\&+\eta'A^2(k)\eta\left[2H_{q+2}\left(\chi_q^2(\alpha);\Delta^2\right)-H_{q+4}\left(\chi_q^2(\alpha);\Delta^2\right)\right]\\&-2\beta'\left(A(k)-I\right)A(k)\eta H_{q+2}\left(\chi_q^2(\alpha);\Delta^2\right)+\beta'\left(A(k)-I\right)^2\beta \quad (17)\end{aligned}$$

Where $R=C^{-1}H'\left(HC^{-1}H'\right)^{-1}HC^{-1}$.

3.1 MSE analysis as a function of Δ

By (17), we have

$$Risk\left(\hat{\beta}_n^{PT}\right)=\sigma_{zz}tr\left(C^{-1}\right)-\sigma_{zz}tr(R)H_{q+2}\left(\chi_q^2(\alpha);\Delta^2\right) \quad (18)$$

Consider the following difference:

$$
\begin{aligned}
& Risk\left(\hat{\beta}_n^{PT}\right)-Risk\left(\hat{\beta}_{nAURE}^{PT}(k)\right) \\
&=\sigma_{zz}tr\left(C^{-1}\left(I-A^2(k)\right)\right)-\sigma_{zz}tr\left(R\left(I-A^2(k)\right)\right)H_{q+2}\left(\chi_q^2(\alpha);\Delta^2\right) \\
&-\eta' A^2(k)\eta\left[2H_{q+2}\left(\chi_q^2(\alpha);\Delta^2\right)-H_{q+4}\left(\chi_q^2(\alpha);\Delta^2\right)\right] \\
&+2\beta'\left(A(k)-I\right)A(k)\eta H_{q+2}\left(\chi_q^2(\alpha);\Delta^2\right)-\beta'\left(A(k)-I\right)^2\beta
\end{aligned} \tag{19}
$$

Observe that $I-A^2(k)=\left(I-A(k)\right)\left(I+A(k)\right)=k^2\left(I_p+kC\right)^{-2}\left(I+A(k)\right)>0$, so when $\Delta\ge\Delta_1$ preliminary test almost unbiased ridge estimator is better than preliminary test estimator, where

$$\Delta_1=\frac{f(k)}{\left[2H_{q+2}\left(\chi_q^2(\alpha);\Delta^2\right)-H_{q+4}\left(\chi_q^2(\alpha);\Delta^2\right)\right]\lambda_p\left(C^{-1}\left(I-A^2(k)\right)\right)}$$

$$
\begin{aligned}
f(k)=&\sigma_{zz}tr\left(R\left(I-A^2(k)\right)\right)H_{q+2}\left(\chi_q^2(\alpha);\Delta^2\right)+k^4\beta'\left(I_p+kC\right)^{-2}\beta \\
&+2k^2\beta'\left(I_p+kC\right)^{-2}A(k)\eta H_{q+2}\left(\chi_q^2(\alpha);\Delta^2\right)-\sigma_{zz}tr\left(C^{-1}\left(I-A^2(k)\right)\right)
\end{aligned}
$$

When $\Delta\le\Delta_2$, preliminary test estimator is better than preliminary test almost unbiased ridge estimator, where

$$\Delta_2=\frac{f(k)}{\left[2H_{q+2}\left(\chi_q^2(\alpha);\Delta^2\right)-H_{q+4}\left(\chi_q^2(\alpha);\Delta^2\right)\right]\lambda_1\left(C^{-1}\left(I-A^2(k)\right)\right)}$$

So we have the following thoerem

Theorem 1: Under the MSE criterion when $\Delta\ge\Delta_1$ $Risk\left(\hat{\beta}_n^{PT}\right)\ge Risk\left(\hat{\beta}_{nAURE}^{PT}(k)\right)$; when $0\le\Delta\le\Delta_2$ $Risk\left(\hat{\beta}_n^{PT}\right)\le Risk\left(\hat{\beta}_{nAURE}^{PT}(k)\right)$.

3.2 MSE analysis as a function of k

When $C>0$, there exists a nonorthogonal matrix P, such that $P'CP=diag\left(\lambda_1,...,\lambda_p\right)$, so by (17) we have;

$$Risk\left(\hat{\beta}_{nAURE}^{PT}(k)\right)=\sum_{i=1}^{p}\frac{t_i^2k^4+k^2\left(\lambda_i+2k\right)\lambda_i g_i+\lambda_i\left(\lambda_i+2k\right)^2\left(\sigma_{zz}-f_i\right)}{\left(\lambda_i+k\right)^4} \tag{20}$$

Where $t=P'\beta$, $a_{ii}=diag\left(P'RP\right)$, $\tilde{\eta}=P'\eta$.

$$f_i=\sigma_{zz}a_{ii}\lambda_i H_{q+2}\left(\chi_q^2(\alpha);\Delta^2\right)-\tilde{\eta}_i^2\left[2H_{q+2}\left(\chi_q^2(\alpha);\Delta^2\right)-H_{q+4}\left(\chi_q^2(\alpha);\Delta^2\right)\right]$$

$$g_i=2t_i\tilde{\eta}_i H_{q+2}\left(\chi_q^2(\alpha);\Delta^2\right)$$

Differentiating the risk function of (20) with respect to k:

$$\frac{\partial Risk\left(\hat{\beta}_{nAURE}^{PT}(k)\right)}{\partial k}=\sum_{i=1}^{p}\frac{h_i(k)}{\left(\lambda_i+k\right)^5} \tag{21}$$

Where $h_i(k)=2\lambda_i k\left\{\left(2t_i^2-g_i\right)k^2+2\left[\lambda_i g_i-2\left(\sigma_{zz}-f_i\right)k\right]+\lambda_i\left[\lambda_i g_i-2\left(\sigma_{zz}-f_i\right)\right]\right\}$.

When $0<k<k_1$, $\frac{\partial Risk\left(\hat{\beta}_{nAURE}^{PT}(k)\right)}{\partial k}<0$, when $k\ge k_2$, $\frac{\partial Risk\left(\hat{\beta}_{nAURE}^{PT}(k)\right)}{\partial k}\ge 0$. Where

$$k_1=\min_{1\le i\le p}\left\{\frac{\sqrt{2q_i\left[\lambda_i\left(g_i-t_i^2\right)-\left(\sigma_{zz}-f_i\right)\right]}-\left[\lambda_i g_i-2\left(\sigma_{zz}-f_i\right)\right]}{2t_i-g_i}\right\}$$

$$k_2 = \max_{1 \le i \le p} \left\{ \frac{\sqrt{2q_i\left[\lambda_i\left(g_i - t_i^2\right) - \left(\sigma_{zz} - f_i\right)\right]} - \left[\lambda_i g_i - 2\left(\sigma_{zz} - f_i\right)\right]}{2t_i - g_i} \right\}$$

So we have;

Theorem 2: Under the MSE criterion, when $0 < k < k_1$, preliminary test almost unbiased ridge estimator is better than preliminary test estimator. When $k \ge k_2$, preliminary test estimator is better than preliminary test almost unbiased ridge estimator.

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