The Discussion of Positive Solution on Partial Differential Equation

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Abstract

By restricting the condition, the paper put forward and proved the non-existence of positive solution for two classes of partial differential equation.

Keywords

Partial differential equation,positive solution,non-existence.

1. Introduction

In the last decades, more and more importance has been attached to the research of partial differential equation. On one aspect ,because plenty of problems involved in these equations originated from many mathematical models in physics、chemistry and biology. Besides, many branches inside mathematics have deep relationship with partial differential equation which as a result has a strong practical application background.

First, we would research oval equation as following

$$\begin{cases} \Delta u + k(|x|)u^p + f(x) = 0 \\ u > 0, x \in R^n \end{cases} \tag{1}$$

Where $n \geq 3, p > 1,$ $f(x) \geq 0$ and $k(|x|) > 0$ are both part of Holder continuous functions given in $R^n \setminus \{0\}$, and $k(|x|) > 0$ satisfies slow decline condition, that is, there exist constant $c > 0$ and $l > -2$ which can result in $k(|x|) > c|x|^l$ when $|x|$ is sufficiently large. If there exists a positive solution u which satisfies the equation (1) at the point of $R^n \setminus \{0\}$, then we call u is a positive solution of the equation (1). Since 1996, G.Bemard and S.Bae have done much research on the $k(|x|) \equiv 1$ case of equation (1) in [1] and [2] and have achieved excellent results. Many troubles still remain when $k(|x|)$ is a nonconstant one. However the $k(|x|)$ slow decline condition used in the paper has given a good discussion about the nonexistence of positive solution for the equation (1).

Then, we would discuss equation as following

$$\begin{cases} \Delta u + |x|^m (u^p - u) = 0 \\ u > 0, x \in R^n \end{cases} \tag{2}$$

To the existence of radial positive solution for the Dirichlet problem in equation (2), [3] has discussed about the case of $p = 3$ and [4] has discussed about the symmetry of solutions in global domain. To be more general, the author has discussed about the nonexistence of positive solution for the Dirichlet problem in the equation in bounded sphere domain. Furthermore, it generalized the conclusions of [4].

2. The Non-Existence of Positive Solution for Equation (1)

Definition 1 Function $\bar{u}$ is called the spherical average function of u .Suppose

$$\bar{u}(r)=\frac{1}{w_n r^{n-1}}\int_{|x|=r}u(x)ds=\frac{1}{w_n}\int_{|\xi|=1}u(r\xi)ds \qquad r>0 \quad \bar{u}(0)=0 \tag{3}$$

w_n is the superficial area of unit sphere, and ds is the surface measure of unit sphere in R^n .

Lemma 1 Let $f(x)\geq 0$ and $k(|x|)>0$ satisfies slow decline condition, that is, there exist constant $c>0$ and $l>-2$ which can make $k(|x|)>c|x|^l$ when $|x|$ is sufficiently large. If u is a positive solution of equation (1), then we have the reckon as following

$0<r^m\bar{u}(r)\leq c$.

Proof We can obtain

$$\Delta\bar{u}+k\bar{u}^p\leq -\bar{f}\leq 0 \tag{4}$$

through the equation (1). We have integral for (4) in B_R . According to Green formula, we get

$$\int_{B_R}k\bar{u}^p dx\leq -\int_{B_R}\Delta\bar{u}dx\leq -\int_{\partial B_R}\bar{u}'ds=-cR^{n-1}\bar{u}'(R) \tag{5}$$

c is a positive constant. Through another usage of inequality (4), we can obtain

$$\left(r^{n-1}\bar{u}'(r)\right)'\leq -r^{n-1}k\bar{u}^p\leq 0$$

Because of $\bar{u}'(0)=0$, $\bar{u}'(r)\leq 0$ is constantly established to $r\geq 0$. Suppose $k(r)\geq cr'$ is constantly established to $r\geq R_0$, then when $R\geq 2R_0$, we get

$$\int_{B_R}k\bar{u}^p dx\geq\int_{B_R\setminus B_{R_0}}k\bar{u}^p dx\geq\int_{B_R\setminus B_{R_0}}cr^l\bar{u}^p dx\geq -c\bar{u}^p(R) \qquad \int_{R_0}^{R}r^{n+l-1}dr\geq cR^{n+1}\bar{u}^p(R) \tag{6}$$

Combine (5) and (6), we get

$$-\frac{\bar{u}'(R)}{\bar{u}^p(R)}\geq cR^{l+1}, R\geq R_1=2R_0 \tag{7}$$

When r is sufficiently large, we have integral for (7) in $\left[R_1, r\right]$,

$$-\int_{R_1}^{r}\frac{\overline{u}'(R)}{\overline{u}^{p}(R)}dR=\frac{\overline{u}^{1-p}(R)}{1-P}\Big|_{R_1}^{r}=\frac{1}{P-1}\left(\overline{u}^{1-p}(r)-\overline{u}^{1-p}(R_1)\right)$$

So when r is sufficiently large, we obtain

$$\overline{u}^{1-p}(r)\geq cr^{2+l}+\overline{u}^{1-p}(R_1)\geq cr^{2+l}>0$$

The lemma is proved right.

Theorem 1 Set $p>\frac{n+l}{n-2}$, $f(x)\geq 0$ and

$$0<k_{\infty}=\lim_{r\to\infty}r^{-l}k(r)\leq r^{-l}k(r)\leq k_0\left(r>0, l>-2\right) \tag{8}$$

If the sphere $\overline{f}$ of function $f(x)$ satisfies

$$\int_0^{\infty}\frac{r^{\frac{2p+l}{p-1}}\overline{f}(r)-\frac{p-1}{\left(p^{p}r^{-l}k\right)^{\frac{1}{p-1}}}L^{p}}{r}dr=+\infty \tag{9}$$

then equation (1) does not have any positive solution in ∞.

Proof set u is a positive solution of equation (1), then the spherical average function $\overline{f}$ of $f(x)$ satisfies

$$\Delta\overline{u}+k\overline{u}^{p}+\overline{f}\leq 0$$

According to the lemma 1, $r^{\frac{2p+l}{p-1}}\overline{u}$ is bounded in ∞. Let

$$v(t)=r^{\frac{2p+l}{p-1}}\overline{u}, r=e^{t}, t\in(-\infty,+\infty)$$

So $v(t)$ satisfies

$$v''(t)+b_0v'(t)-L^{p-1}v(t)+k(t)v^{p}(t)+g(t)\leq 0$$
$$t\in(-\infty,+\infty)$$

where

$$b_0=n-2-2m, L=\left[m(n-2-m)\right]^{\frac{1}{p-1}}$$

$$k(t)=r^{-l}k(r), g(t)=r^{\frac{2p+l}{p-1}}\overline{f}(r)$$

If $k(r)$ satisfies the condition given by the problem set, then according to

$$\max_{v\in(0,+\infty)}\left(L^{p-1}v-k(t)v^{p}\right)=\frac{p-1}{\left(p^{p}k(t)\right)^{\frac{1}{p-1}}}L^{p}$$

and

$$v''(t)+b_0v'(t)-\frac{p-1}{\left(p^{p}k(t)\right)^{\frac{1}{p-1}}}L^{p}+g(t)\leq 0 \tag{10}$$

We have integral for (10) and can get

$$v'(t) \le v'(T) - b_0\left(v(t) - v(T)\right) - \int_T^t \left[g(t) - \frac{p-1}{\left(p^p k(t)\right)^{\frac{1}{p-1}}} L^p \right] dt$$

Now that $v(t)$ is bounded when $t \ge T$, then we can get $v'(t) < 0$ through the combination of (9) and the inequality above when t is sufficiently larger than T . So the contradiction $v = 0$ appears at the point of $t_0 \ge T$. Thus, the theorem 1 is proved right.

3. The Non-Existence of Positive Solution for Equation (2)

Lemma 2 set Ω is the bounded smooth area in R^n , and u is the classical solution for the equation $\Delta u + f(x,u) = 0$

then we obtain Rellich-Pohozaev identity with $n \ge 3$:

$$\begin{aligned} &\int_\Omega \left[nF(x,u) - \frac{n-2}{2} u f(x,u) + F_x(x,u) \right] dx \\ &= \int_{\partial\Omega} \left[(x \cdot \nabla u)\frac{\partial u}{\partial n} - (x \cdot n)\frac{|\nabla u|^2}{2} + (x \cdot n) F(x,u) + \frac{n-2}{2} u \frac{\partial u}{\partial n} \right] ds \end{aligned} \tag{11}$$

where $F(x,u) = \int_0^u f(x,t)dt$, and $F_x(x,u)$ is the gradient of $F(x,u)$ which alters with x , and n is the unit outside normal vector of $\partial\Omega$, and ds is the area element on $\partial\Omega$.

Proof Set

$$V(x) = (x \cdot \nabla u)\nabla u - \frac{|\nabla u|^2}{2} x + xF(x,u) + \frac{n-2}{2} u \nabla u$$

Use the equation which u satisfies, we can calculate directly to get

$$divV(x) = nF(x,u) - \frac{n-2}{2} u f(x,u) + xF_x(x,u)$$

According to the divergence theorem

$$\int_\Omega divV(x)dx = \int_{\partial\Omega} V(x) \cdot n ds ,$$

we can get (11)

Theorem 2 We discuss equation (2) as following

$$\begin{cases} \Delta u + |x|^m \left(u^p - u\right) = 0 & in \quad B_b(0) \\ u > 0, x \in R^n & in \quad B_b(0) \end{cases}$$

When

$$p \ge \frac{2m+n+2}{n-2}$$

equation (2) does not have any solution. In this equation,

$B_b(0)$ stands for the sphere domain whose center of sphere is at the origin and radius is b .

Proof To equation (2), we can get

$$f(x,u) = |x|^m \left(u^p - u\right)$$

$$F(x,u) = \int_0^u |x|^m (t^p - t)\, dt$$
$$= |x|^m \left(\frac{u^{p+1}}{p+1} - \frac{u^2}{2} \right)$$

and

$$F(x,u)\Big|_{\partial B_b(0)} = 0$$

as well as

$$u\Big|_{\partial B_b(0)} = 0$$

according to lemma 2.

Substitute these above into (11), we can get

$$\int_{B_b(0)} \left(\frac{m+n}{p+1} - \frac{n-2}{2} \right) u^{p+1} dx = \int_{B_b(0)} \left(\frac{2n+m-2}{2} \right) u^2 dx + \int_{\partial B_b(0)} \frac{1}{2} (x \cdot n) |\nabla u|^2\, dx \tag{12}$$

Since the vector x at any point at the boundary of sphere domain $B_b(0)$ has the same direction with the outside normal vector at that point, we can know $(x \cdot n) > 0$, as a result, the right side of (12) is greater than zero. But when

$$p \ge \frac{2m+n+2}{n-2}$$

inequality

$$\left(\frac{m+n}{p+1} - \frac{n-2}{2} \right) \le 0$$

leads the lest of (12) to be less than zero, which is the contradiction with Rellich-Pohozaev identity. So theorem 2 is proved right.

Notice that sphere domain is used in the proof of theorem 2, we can generalize common starlike domains with the favor of theorem 2.

Theorem 2 To the equation as following

$$\begin{cases} \Delta u + |x|^m (u^p - u) = 0 & in \quad \Omega \\ u > 0, x \in R^n & in \quad \Omega \end{cases}$$

When

$$p \ge \frac{2m+n+2}{n-2}$$

the equation does not have any solution.

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