Immediate probability hesitant fuzzy Hamacher ordered weighted geometric operator

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Abstract

In this paper, we investigate the decision making problems by using probabilities, immediate probabilities and information that can be represented with hesitant fuzzy information. Then, we have developed some new probability geometric aggregation operators with hesitant fuzzy information: probability hesitant fuzzy weighted geometric (P-HFWG) operator, immediate probability hesitant fuzzy ordered weighted geometric (IP-HFOWG) operator and probability hesitant fuzzy ordered weighted geometric (P-HFOWG) operator. Furthermore, we shall develop some hesitant fuzzy Hamacher aggregation operator with immediate probabilities. These operators include: probability hesitant fuzzy Hamacher weighted geometric (P-HFHWG) operator, immediate probability hesitant fuzzy Hamacher ordered weighted geometric(IP-HFOWG) operator and probability hesitant fuzzy Hamacher ordered weighted geometric (P-HFOWG) operator.

Keywords

decision making; hesitant fuzzy set; operational laws; probabilistic geometric aggregating operators; Hamacher operations.

1. Introduction

In 2010, Torra [1] proposed the hesitant fuzzy set which permits the membership having a set of possible values and discussed the relationship between hesitant fuzzy set and intuitionistic fuzzy set, and showed that the envelope of hesitant fuzzy set is an intuitionistic fuzzy set. Xia and Xu [2] gave an intensive study on hesitant fuzzy information aggregation techniques and their application in decision making. Xu et al. [3] developed several series of aggregation operators for hesitant fuzzy information with the aid of quasi-arithmetic means. Gu et al. [4] utilized the hesitant fuzzy weighted averaging (HFWA) operator to investigat the evaluation model for risk investment with hesitant fuzzy information. Motivated by the ideal of prioritized aggregation operators [5], Wei [6] developed some prioritized aggregation operators for aggregating hesitant fuzzy information, and then apply them to develop some models for hesitant fuzzy multiple attribute decision making (MADM) problems in which the attributes are in different priority level. Wei et al. [7] proposed two hesitant fuzzy Choquet integral aggregation operators: hesitant fuzzy choquet ordered averaging (HFCOA) operator and hesitant fuzzy choquet ordered geometric (HFCOG) operator and applied the HFCOA and HFCOG operators to multiple attribute decision making with hesitant fuzzy information. Furthermore, they proposed the generalized hesitant fuzzy choquet ordered averaging (GHFCOA) operator and generalized hesitant fuzzy choquet ordered geometric (GHFCOG) operator. Zhu et al. [8] explored the geometric Bonferroni mean (GBM) considering both the BM and the geometric mean (GM) under hesitant fuzzy environment. They further defined the hesitant fuzzy geometric Bonferroni mean (HFGBM) and the hesitant fuzzy Choquet geometric Bonferroni mean (HFCGBM). Then they gave the definition of hesitant fuzzy geometric Bonferroni element (HFGBE), which is considered as the basic calculational unit in the HFGBM and reflects the conjunction between two aggregated arguments. The properties and special cases of the HFGBM are studied in detail based on the discussion of the HFGBE. In addition, the weighted hesitant fuzzy geometric Bonferroni mean (WHFGBM) and the weighted hesitant fuzzy Choquet geometric Bonferroni mean (WHFCGBM) are proposed considering the importance of each argument and the correlations among them.

In this paper, we investigate the decision making problems by using probabilities, immediate probabilities and information that can be represented with hesitant fuzzy information. Then, we have developed some new probability geometric aggregation operators with hesitant fuzzy information: probability hesitant fuzzy weighted geometric (P-HFWG) operator, immediate probability hesitant fuzzy ordered weighted geometric (IP-HFOWG) operator and probability hesitant fuzzy ordered weighted geometric (P-HFOWG) operator. Furthermore, we shall develop some hesitant fuzzy Hamacher aggregation operator with immediate probabilities. These operators include: probability hesitant fuzzy Hamacher weighted geometric (P-HFHWG) operator, immediate probability hesitant fuzzy Hamacher ordered weighted geometric (IP-HFOWG) operator and probability hesitant fuzzy Hamacher ordered weighted geometric (P-HFOWG) operator.

2. Preliminaries

In the following, we shall introduce some basic concepts related to hesitant fuzzy sets and immediate probabilities.

2.1 Hesitant fuzzy sets

Atanassov[9] extended the fuzzy set to the IFS. However, when giving the membership degree of an element, the difficulty of establishing the membership degree is not because we have a margin of error, or some possibility distribution on the possibility values, but because we have several possible values. For such cases, Torra[1] proposed another generation of FS.

Definition 1 [1]**.** Given a fixed set X , then a hesitant fuzzy set (HFS) on X is in terms of a function that when applied to X returns a sunset of $\left[0,1\right]$.

To be easily understood, Xu express the HFS by mathematical symbol:

$$E=\left(\left\langle x,h_E\left(x\right)\right\rangle\middle|x\in X\right), \tag{1}$$

where $h_E\left(x\right)$ is a set of some values in $\left[0,1\right]$, denoting the possible membership degree of the element $x\in X$ to the set E . For convenience, Xu call $h=h_E\left(x\right)$ a hesitant fuzzy element(HFE) and H the set of all HFEs.

Definition 2[2]**.** For a HFE h, $s\left(h\right)=\frac{1}{\#h}\sum_{\gamma\in h}\gamma$ is called the score function of h, where $\#h$ is the number of the elements in h . For two HFEs h_1 and h_2 , if $s\left(h_1\right)>s\left(h_2\right)$, then $h_1>h_2$; if $s\left(h_1\right)=s\left(h_2\right)$, then $h_1=h_2$.

Based on the relationship between the HFEs and IFVs, Xia and Xu[2] define some new operations on the HFEs h , h_1 and h_2 :

$h^{\lambda}=\bigcup_{\gamma\in h}\left\{\gamma^{\lambda}\right\}$;

(2) $\lambda h=\bigcup_{\gamma\in h}\left\{1-\left(1-\gamma\right)^{\lambda}\right\}$;

(3) $h_1\oplus h_2=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2}\left\{\gamma_1+\gamma_2-\gamma_1\gamma_2\right\}$;

(4) $h_1\otimes h_2=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2}\left\{\gamma_1\gamma_2\right\}$.

2.2 Hamacher operations

T-norm and t-conorm are an important notion in fuzzy set theory, which are used to define a generalized union and intersection of fuzzy sets [10]. Roychowdhury and Wang [11] gave the definition and conditions of t-norm and t-conorm. Based on a t-norm (T) and t-conorm (T^*), a generalized union and a generalized intersection of intuitionistic fuzzy sets were introduced by Deschrijver and Kerre [12]. Further, Hamacher[13] proposed a more generalized t-norm and t-conorm. Hamacher operation[13] include the Hamacher product and Hamacher sum, which are examples of t-norms and t-conorms, respectively. They are defined as follows:

Hamacher product $\otimes$ is a t-norm and Hamacher sum $\oplus$ is a t-conorm, where

$$T\left(a,b\right)=a\otimes b=\frac{ab}{\gamma+\left(1-\gamma\right)\left(a+b-ab\right)} \tag{2}$$

$$T^*\left(a,b\right)=a\oplus b=\frac{a+b-ab-(1-\gamma)ab}{1-(1-\gamma)ab} \tag{3}$$

Especially, when $\gamma=1$, then Hamacher t-norm and t-conorm will reduce to

$$T\left(a,b\right)=a\otimes b=ab \tag{4}$$

$$T^*\left(a,b\right)=a\oplus b=a+b-ab \tag{5}$$

which are the algebraic t-norm and t-conorm respectively; when $\gamma=2$, then Hamacher t-norm and t-conorm will reduce to

$$T\left(a,b\right)=a\otimes b=\frac{ab}{1+\left(1-a\right)\left(1-b\right)} \tag{6}$$

$$T^*\left(a,b\right)=a\oplus b=\frac{a+b}{1+ab} \tag{7}$$

Which are called the Einstein t-norm and t-conorm respectively[14].

2.3 Probabilistic geometric aggregating operators

Merigó[15-16] developed the probabilistic weighted geometric (PWG) operator which is an aggregation operator that unifies the probability and the weighted geometric in the same formulation considering the degree of importance that each concept has in the aggregation.

Definition 3. An PWG operator of dimension n is a mapping PWG: $R^n \to R$, such that

$$\mathrm{PWG}\left(a_1,a_2,\cdots,a_n\right)=\prod_{j=1}^{n}\left(a_j\right)^{v_j} \tag{8}$$

where $\omega=\left(\omega_1,\omega_2,\cdots,\omega_n\right)^T$ be the weight vector of $a_j\left(j=1,2,\cdots,n\right)$, and $\omega_j>0$, $\sum_{j=1}^{n}\omega_j=1$, and a probabilistic weight $p_j>0$, $\sum_{j=1}^{n}p_j=1$, $v_j=\beta p_j+\left(1-\beta\right)\omega_j$ with $\beta\in\left[0,1\right]$ and v_j is the weight that unifies probabilities and WGs in the same formulation, then PWG is called the probabilistic weighted geometric (PWG) operator. The PWG operator is monotonic, commutative, bounded and idempotent[15-16].

In order to develop the analysis, Merigó[17] used in the same formulation the weights of the OWG operator and the probabilistic information and proposed the immediate probability OWG(IP-OWg) operator. It can be defined as follows.

Definition 4[17]**.** An IP-OWG operator of dimension n is a mapping IP-OWG: $R^n\rightarrow R$, that has an associated weight vector $w=\left(w_1,w_2,\cdots,w_n\right)^T$ such that $w_j>0$ and $\sum_{j=1}^{n}w_j=1$. Furthermore,

$$\text{IP-OWG}\left(a_1,a_2,\cdots,a_n\right)=\prod_{j=1}^{n}\left(a_{\sigma(j)}\right)^{\hat{p}_j} \tag{9}$$

where $\left(\sigma(1),\sigma(2),\cdots,\sigma(n)\right)$ is a permutation of $\left(1,2,\cdots,n\right)$, such that $\alpha_{\sigma(j-1)}\geq\alpha_{\sigma(j)}$ for all $j=2,\cdots,n$, each a_i has associated a probability p_i, p_j is the associated probability of $\alpha_{\sigma(j)}$, and $\hat{p}_j=\dfrac{w_jp_j}{\sum_{j=1}^{n}w_jp_j}$.

It's worth pointing out that IP-OWG operator is a good approach for unifying probabilities and OWGs in some particular situations. But it is not always useful, especially in situations where we want to give more importance to the probabilities or to the OWG operators. In order to show why this unification does not seem to be a final model, we could also consider other ways of representing $\hat{p}_j$.

For example, we could also use $\hat{p}_j=\dfrac{w_j+p_j}{\sum_{j=1}^{n}\left(w_j+p_j\right)}$ or other similar approaches.

Another approach for unifying probabilities and OWGs in the same formulation is the probabilistic OWG (POWG) operator [15-20]. Its main advantage is that it is able to include both concepts considering the degree of importance of each case in the problem.

Definition 5. An POWG operator of dimension n is a mapping PWG: $R^n\rightarrow R$ that has an associated weight vector $w=\left(w_1,w_2,\cdots,w_n\right)^T$ such that $w_j>0$ and $\sum_{j=1}^{n}w_j=1$, according to the following formula:

$$\text{POWG}\left(a_1,a_2,\cdots,a_n\right)=\prod_{j=1}^{n}\left(a_{\sigma(j)}\right)^{\hat{p}_j} \tag{10}$$

where $(\sigma(1),\sigma(2),\cdots,\sigma(n))$ is a permutation of $(1,2,\cdots,n)$, such that $\alpha_{\sigma(j-1)} \geq \alpha_{\sigma(j)}$ for all $j=2,\cdots,n$, each a_i has associated a probability p_i, with $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0,1]$, $\hat{p}_j = \beta p_j + (1-\beta) w_j$ with $\beta \in [0,1]$ and p_j is the associated probability of $\alpha_{\sigma(j)}$.

3. Hesitant fuzzy geometric aggregating operator with immediate probabilities

Based on the aggregation principle for HFEs, Xia and Xu [2] developed the hesitant fuzzy weighted geometric(HFWG) operator and hesitant fuzzy ordered weighted geometric (HFOWG) operator.

Definition 6. Let $h_j\,(j=1,2,\cdots,n)$ be a collection of HFEs. The hesitant fuzzy weighted geometric (HFWG) operator is a mapping $H^n \to H$ such that

$$\mathrm{HFWG}(h_1,h_2,\cdots,h_n) = \otimes_{j=1}^{n} (h_j)^{\omega_j} = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \cdots, \gamma_n \in h_n} \left\{ \prod_{j=1}^{n} (\gamma_j)^{\omega_j} \right\} \tag{11}$$

where $\omega = (\omega_1,\omega_2,\cdots,\omega_n)^T$ be the weight vector of $h_j\,(j=1,2,\cdots,n)$, and $\omega_j > 0$, $\sum_{j=1}^{n} \omega_j = 1$.

Definition 7[2]**.** Let $h_j\,(j=1,2,\cdots,n)$ be a collection of HFEs, the hesitant fuzzy ordered weighted geometric (HFOWG) operator of dimension n is a mapping HFOWG: $H^n \to H$, that has an associated weight vector $w = (w_1, w_2, \cdots, w_n)^T$ such that $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$. Furthermore,

$$\begin{aligned} \mathrm{HFOWG}(h_1,h_2,\cdots,h_n) &= \otimes_{j=1}^{n} \left(h_{\sigma(j)}\right)^{w_j} \\ &= \bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \cdots, \gamma_{\sigma(n)} \in h_{\sigma(n)}} \left\{ \prod_{j=1}^{n} \left(\gamma_{\sigma(j)}\right)^{w_j} \right\} \end{aligned} \tag{12}$$

where $(\sigma(1),\sigma(2),\cdots,\sigma(n))$ is a permutation of $(1,2,\cdots,n)$, such that $h_{\sigma(j-1)} \geq h_{\sigma(j)}$ for all $j=2,\cdots,n$.

Similarly, the PWA operator, IPOWA operator and POWA operator have usually been used in situations where the input arguments are the exact values. In the following, we shall extend these operators to accommodate the situations where the input arguments are hesitant fuzzy information. These operators include: probability hesitant fuzzy weighted average (P-HFWA) operator, immediate probability hesitant fuzzy ordered weighted average (IP-HFOWA) operator and probability hesitant fuzzy ordered weighted average (P-HFOWA) operator.

Definition 8. Let $h_j\,(j=1,2,\cdots,n)$ be a collection of HFEs, an P- HFWG operator of dimension n is a mapping P- HFWG: $Q^n \to Q$, such that

$$\text{P-HFWG}_{\hat{v}}(h_1,h_2,\cdots,h_n) = \otimes_{j=1}^{n} (h_j)^{\hat{v}_j} = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \cdots, \gamma_n \in h_n} \left\{ \prod_{j=1}^{n} (\gamma_j)^{\hat{v}_j} \right\} \tag{13}$$

where $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$ be the weight vector of $a_j\ (j=1,2,\cdots,n)$, and $\omega_j>0$, $\sum_{j=1}^{n}\omega_j=1$, and a probabilistic weight $p_j>0$, $\sum_{j=1}^{n}p_j=1$, $\hat{v}_j=\beta p_j+(1-\beta)\omega_j$ with $\beta\in[0,1]$ and $\hat{v}_j$ is the weight that unifies probabilities and HFWGs in the same formulation.

Especially, if $\beta=0$, then, the P-HFWG operator reduces to the HFWG operator. And if $\beta=1$, it becomes the hesitant fuzzy probabilistic geometric (HFPG).

The IP-HFOWG operator is an aggregation operator that uses probabilities and OWGs in the same formulation and information represented with hesitant fuzzy information.

Definition 9. Let $h_j\ (j=1,2,\cdots,n)$ be a collection of HFEs, an IP-HFOWG operator of dimension n is a mapping IP-HFOWG: $Q^n\to Q$, that has an associated weight vector $w=(w_1,w_2,\cdots,w_n)^T$ such that $w_j>0$ and $\sum_{j=1}^{n}w_j=1$. Furthermore,

$$\begin{aligned}&\text{IP-HFOWG}_{\hat{p}}(h_1,h_2,\cdots,h_n)=\bigotimes_{j=1}^{n}\left(h_{\sigma(j)}\right)^{\hat{p}_j}\\&=\bigcup_{\gamma_{\sigma(1)}\in h_{\sigma(1)},\gamma_{\sigma(2)}\in h_{\sigma(2)},\cdots,\gamma_{\sigma(n)}\in h_{\sigma(n)}}\left\{\prod_{j=1}^{n}\left(\gamma_{\sigma(j)}\right)^{\hat{p}_j}\right\}\end{aligned}\tag{14}$$

where $(\sigma(1),\sigma(2),\cdots,\sigma(n))$ is a permutation of $(1,2,\cdots,n)$, such that $h_{\sigma(j-1)}\geq h_{\sigma(j)}$ for all $j=2,\cdots,n$, each a_i has associated a probability p_i, p_j is the associated probability of $\tilde{\alpha}_{\sigma(j)}$, and $\hat{p}_j=\dfrac{w_jp_j}{\sum_{j=1}^{n}w_jp_j}$.

It's worth pointing out that IP-HFOWG operator is a good approach for unifying probabilities and HFOWGs in some particular situations. But it is not always useful, especially in situations where we want to give more importance to the probabilities or to the HFOWG operators. In order to show why this unification does not seem to be a final model, we could also consider other ways of representing $\hat{p}_j$. For example, we could also use $\hat{p}_j=\dfrac{w_j+p_j}{\sum_{j=1}^{n}(w_j+p_j)}$ or other similar approaches.

The P-HFOWG operator unifies the probability and the OWG operator in the same formulation considering the degree of importance of each concept in the aggregation. It also uses information represented in the form of hesitant fuzzy information.

Definition 10. Let $h_j\ (j=1,2,\cdots,n)$ be a collection of HFEs, an P-HFOWG operator of dimension n is a mapping P-HFOWG: $Q^n\to Q$, that has an associated weight vector $w=(w_1,w_2,\cdots,w_n)^T$ such that $w_j>0$ and $\sum_{j=1}^{n}w_j=1$. Furthermore,

$$\text{P-HFOWG}_{\tilde{p}}\left(h_1,h_2,\cdots,h_n\right)=\overset{n}{\underset{j=1}{\otimes}}\left(h_{\sigma(j)}\right)^{\tilde{p}_j}$$
$$=\bigcup_{\gamma_{\sigma(1)}\in h_{\sigma(1)},\gamma_{\sigma(2)}\in h_{\sigma(2)},\cdots,\gamma_{\sigma(n)}\in h_{\sigma(n)}}\left\{\prod_{j=1}^{n}\left(\gamma_{\sigma(j)}\right)^{\tilde{p}_j}\right\} \tag{15}$$

where $\left(\sigma(1),\sigma(2),\cdots,\sigma(n)\right)$ is a permutation of $(1,2,\cdots,n)$, such that $h_{\sigma(j-1)}\geq h_{\sigma(j)}$ for all $j=2,\cdots,n$, each $\tilde{a}_i$ has associated a probability p_i, with $\sum_{i=1}^{n}p_i=1$ and $p_i\in[0,1]$, $\tilde{p}_j=\beta p_j+(1-\beta)w_j$ with $\beta\in[0,1]$ and p_j is the associated probability of $\tilde{\alpha}_{\sigma(j)}$.

Especially, if $\beta=0$, then, the P-HFOWG operator reduces to the HFOWG operator. And if $\beta=1$, it becomes the hesitant fuzzy probabilistic geometric (HFPG) operator.

4. Hesitant fuzzy Hamacher geometric aggregating operator with immediate probabilities

Motivated by the arithmetic aggregation operators[21-26], the Hamacher product $\otimes$ and the Hamacher sum $\oplus$, then the generalized intersection and union on two HFEs h_1 and h_2 become the Hamacher product(denoted by $h_1\otimes h_2$) and Hamacher sum(denoted by $h_1\oplus h_2$) of two HFEs h_1 and h_2, respectively, as follows:

(1) $h_1\oplus h_2=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2}\left\{\dfrac{\gamma_1+\gamma_2-\gamma_1\gamma_2-(1-\gamma)\gamma_1\gamma_2}{1-(1-\gamma)\gamma_1\gamma_2}\right\}$;

(2) $h_1\otimes h_2=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2}\left\{\dfrac{\gamma_1\gamma_2}{\gamma+(1-\gamma)\left(\gamma_1+\gamma_2-\gamma_1\gamma_2\right)}\right\}$.

(3) $\lambda h_1=\bigcup_{\gamma_1\in h_1}\left\{\dfrac{\left(1+(\gamma-1)\gamma_1\right)^{\lambda}-\left(1-\gamma_1\right)^{\lambda}}{\left(1+(\gamma-1)\gamma_1\right)^{\lambda}+(\gamma-1)\left(1-\gamma_1\right)^{\lambda}}\right\},\lambda>0$;

(4) $\left(h_1\right)^{\lambda}=\bigcup_{\gamma_1\in h_1}\left\{\dfrac{\gamma\left(\gamma_1\right)^{\lambda}}{\left(1+(\gamma-1)(1-\gamma_1)\right)^{\lambda}+(\gamma-1)\left(\gamma_1\right)^{\lambda}}\right\},\lambda>0$.

In the following, we shall develop some hesitant fuzzy Hamacher aggregation operator with immediate probabilities. These operators include: probability hesitant fuzzy Hamacher weighted geometric (P-HFHWG) operator, immediate probability hesitant fuzzy Hamacher ordered weighted geometric (IP-HFOWG) operator and probability hesitant fuzzy Hamacher ordered weighted geometric (P-HFOWG) operator.

Definition 11. Let $h_j\left(j=1,2,\cdots,n\right)$ be a collection of HFEs, an P- HFHWG operator of dimension n is a mapping P- HFHWG: $Q^n\rightarrow Q$, such that

$$\text{P-HFHWG}_{\hat{v},\lambda}\left(h_1,h_2,\cdots,h_n\right)=\overset{n}{\underset{j=1}{\otimes}}\left(h_j\right)^{\hat{v}_j}$$
$$=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\frac{\lambda\prod_{j=1}^{n}\gamma_j^{\hat{v}_j}}{\prod_{j=1}^{n}\left(1+(\lambda-1)\left(1-\gamma_j\right)\right)^{\hat{v}_j}+(\lambda-1)\prod_{j=1}^{n}\gamma_j^{\hat{v}_j}}\right\},\lambda>0. \tag{16}$$

where $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$ be the weight vector of $a_j\ (j=1,2,\cdots,n)$, and $\omega_j>0$, $\sum_{j=1}^{n}\omega_j=1$, and a probabilistic weight $p_j>0$, $\sum_{j=1}^{n}p_j=1$, $\hat{v}_j=\beta p_j+(1-\beta)\omega_j$ with $\beta\in[0,1]$ and $\hat{v}_j$ is the weight that unifies probabilities and HFHWGs in the same formulation.

The IP-HFHOWG operator is an aggregation operator that uses probabilities and HOWGs in the same formulation and information represented with hesitant fuzzy information.

Definition 12. Let $h_j\ (j=1,2,\cdots,n)$ be a collection of HFEs, an IP-HFHOWG operator of dimension n is a mapping IP-HFHOWG: $Q^n\to Q$, that has an associated weight vector $w=(w_1,w_2,\cdots,w_n)^T$ such that $w_j>0$ and $\sum_{j=1}^{n}w_j=1$. Furthermore,

$$\text{IP-HFHOWG}_{\hat{p},\lambda}(h_1,h_2,\cdots,h_n)=\overset{n}{\underset{j=1}{\otimes}}\left(h_{\sigma(j)}\right)^{\hat{p}_j}$$

$$=\bigcup_{\gamma_{\sigma(1)}\in h_{\sigma(1)},\gamma_{\sigma(2)}\in h_{\sigma(2)},\cdots,\gamma_{\sigma(n)}\in h_{\sigma(n)}}\left\{\frac{\lambda\prod_{j=1}^{n}\gamma_{\sigma(j)}^{\hat{p}_j}}{\prod_{j=1}^{n}\left(1+(\lambda-1)\left(1-\gamma_{\sigma(j)}\right)\right)^{\hat{p}_j}+(\lambda-1)\prod_{j=1}^{n}\gamma_{\sigma(j)}^{\hat{p}_j}}\right\},\lambda>0. \tag{17}$$

where $(\sigma(1),\sigma(2),\cdots,\sigma(n))$ is a permutation of $(1,2,\cdots,n)$, such that $h_{\sigma(j-1)}\geq h_{\sigma(j)}$ for all $j=2,\cdots,n$, each a_i has associated a probability p_i, p_j is the associated probability of $h_{\sigma(j)}$, and $\hat{p}_j=\frac{w_jp_j}{\sum_{j=1}^{n}w_jp_j}$.

The P-HFHOWG operator unifies the probability and the HOWG operator in the same formulation considering the degree of importance of each concept in the aggregation. It also uses information represented in the form of hesitant fuzzy information.

Definition 13. Let $h_j\ (j=1,2,\cdots,n)$ be a collection of HFEs, an P-HFHOWG operator of dimension n is a mapping P-HFHOWG: $Q^n\to Q$, that has an associated weight vector $w=(w_1,w_2,\cdots,w_n)^T$ such that $w_j>0$ and $\sum_{j=1}^{n}w_j=1$. Furthermore,

$$\text{P-HFHOWG}_{\tilde{p},\lambda}(h_1,h_2,\cdots,h_n)=\overset{n}{\underset{j=1}{\otimes}}\left(h_{\sigma(j)}\right)^{\tilde{p}_j}$$

$$=\bigcup_{\gamma_{\sigma(1)}\in h_{\sigma(1)},\gamma_{\sigma(2)}\in h_{\sigma(2)},\cdots,\gamma_{\sigma(n)}\in h_{\sigma(n)}}\left\{\frac{\lambda\prod_{j=1}^{n}\gamma_{\sigma(j)}^{\tilde{p}_j}}{\prod_{j=1}^{n}\left(1+(\lambda-1)\left(1-\gamma_{\sigma(j)}\right)\right)^{\tilde{p}_j}+(\lambda-1)\prod_{j=1}^{n}\gamma_{\sigma(j)}^{\tilde{p}_j}}\right\},\lambda>0 \tag{18}$$

where $(\sigma(1),\sigma(2),\cdots,\sigma(n))$ is a permutation of $(1,2,\cdots,n)$, such that $h_{\sigma(j-1)} \geq h_{\sigma(j)}$ for all $j=2,\cdots,n$, each $\tilde{a}_i$ has associated a probability p_i, with $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0,1]$, $\tilde{p}_j = \beta p_j + (1-\beta) w_j$ with $\beta \in [0,1]$ and p_j is the associated probability of $\tilde{\alpha}_{\sigma(j)}$.

5. Conclusion

In this paper, we investigate the decision making problems by using probabilities, immediate probabilities and information that can be represented with hesitant fuzzy information. Then, we have developed some new probability geometric aggregation operators with hesitant fuzzy information: probability hesitant fuzzy weighted geometric (P-HFWG) operator, immediate probability hesitant fuzzy ordered weighted geometric (IP-HFOWG) operator and probability hesitant fuzzy ordered weighted geometric (P-HFOWG) operator. Furthermore, we shall develop some hesitant fuzzy Hamacher aggregation operator with immediate probabilities. These operators include: probability hesitant fuzzy Hamacher weighted geometric (P-HFHWG) operator, immediate probability hesitant fuzzy Hamacher ordered weighted geometric (IP-HFOWG) operator and probability hesitant fuzzy Hamacher ordered weighted geometric (P-HFOWG) operator.

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