Research on the Endomorphisms of Cycle

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Abstract

In this paper, the half-strong, the locally-strong, the quasi-strong and the strong endomorphisms of cycles are characterized. The endomorphism spectrum and the endomorphism type of a cycle are given.

Keywords

Endomorphism; endomorphism spectrum; graph; endomorphism type.

1. Introduction and Preliminaries

Endomorphism monoid of graphs is a generalization of automorphism group of graphs. In the recent years, much attention has been paid to endomorphism monoids of graph and many interesting results about graphs and their endomorphism monoids have been obtained. The aim of this paper is to establish the relationship between graph theory and algebra theory of semigroup and to apply the theory of semigroups to graph theory. As Petrich and Reilly pointed out in [9], in a wide variety of semigroup theories, the regular semigroup is the central position in the rule of the structure. Therefore, The natural question is: What kind of graph is regular? (the problem is raised by Marki in [8]). The exact answer is also very difficult to get, so the problem is the first step to solve the problem of various types of graph start research. In [10], has found out exactly what the bidirectional connected graph endomorphism monoid is regular. A regular split graph endomorphism monoid has been studied in [6]. This study is the cycle of endomorphism monoid.

The graphs discussed in this paper are all finite simple graphs. Let X be a graph. The vertex set of X is denoted by V (X) and the edge set of X is denoted by E(X). The

Let X and Y be graphs. A mapping f: $V(X)\rightarrow V(Y)$; point set V (X) is called the order number of graph X.

If two vertices x_1 and x_2 are adjacent in a graph X,then the edge connecting x_1 and x_2 is denoted by $\{x_1, x_2\}$ and write $\{x_1, x_2\}\in E(X)$.There is a path of length n with n+1 pionts denoted by Pn, n points of the cycle denoted by Cn.

Definition 1.1 A mapping f is called a homomorphism if $\{x_1, x_2\}\in E(X)$ implies $\{f(x_1), f(x_2)\}\in E(Y)$; Written f$\in$Hom(X,Y)

Definition 1.2 A homomorphism f is called half-strong if $\{f(a), f(b)\}\in E(Y)$

Implies that there exist $x_1, x_2\in V(X)$ with $f(x_1)= f(a)$ and $f(x_2)= f(b)$

such that $\{x_1, x_2\}\in E(X)$; Written f$\in$HHom(X,Y)

Definition 1.3 A homomorphism f is called locallystrong if $\{f(a), f(b)\}\in E(Y)$

Implies that for every preimage $x_1 \in V(X)$ of f(a) there exists a preimage $x_2 \in V(X)$ of f(b) such that $\{x_1, x_2\} \in E(X)$ and analogously for every preimage of f(b); Written f$\in$ LHom(X,Y)

Definition 1.4 A homomorphism f is called quasi-strong if $\{f(a), f(b)\} \in E(Y)$

Implies that there exists a preimage $x_1 \in V(X)$ of f(a) which is adjacent to every preimage of f(b) and analogously for preimage of f(b); Written f$\in$QHom(X,Y)

Definition 1.5 A homomorphism f is called strong if $\{f(a), f(b)\} \in E(Y)$ implies that any preimage of f(a) is adjacent to any preimage of f(b); Written f$\in$SHom(X,Y)

Definition 1.6 A homomorphism f is called an ismorphism if f is bijective and f^{-1} is a homomorphism; Written f$\in$Iso(X,Y)

If X=Y, a homomorphism is called an endomorphism, an ismorphism is called an automorphism. Hom(X,Y), HHom(X,Y), LHom(X,Y), QHom(X,Y), SHom(X,Y), Iso(X,Y) are denoted by End(X), hEnd(X), lEnd(X), QEnd(X), SEnd(X), Aut(X);

we always have

$$End(X) \supseteq HEnd(X) \supseteq LEnd(X) \supseteq QEnd(X) \supseteq SEnd(X) \supseteq Aut(X)$$

It is well known that SEnd(X) forms a monoid with respect to the composition of mapping and End(X) is a monoid and Aut(X) is a group.

Recall from Proposition 2.1 in [3] that HEnd(X), LEnd(X) and QEnd(X) do not form a monoid in general. Various endomorphisms were investigated by many authors (see [3] and its references). To pursue a more systematic treatment of dierent endomorphisms, Bottcher and Knauer in [3] introduced the concepts of the endomorphism spectrum and the endomorphism type of a graph. For a graph X, the 6-tuple

$$\left(|End(X)|, |HEnd(X)|, |LEnd(X)|, |QEnd(X)|, |SEnd(X)|, |Aut(X)|\right)$$

is called the endomorphism spectrum of X and is denoted by Endospec X, that is,

Endospes X = $\left(|End(X)|, |HEnd(X)|, |LEnd(X)|, |QEnd(X)|, |SEnd(X)|, |Aut(X)|\right)$

Associate with Endospec X a 5-tuple $(s_1, s_2, s_3, s_4, s_5)$ with $s_i \in \{0,1\}$, i = 1,2,3,4,5, where $s_i = 0$ indicates that the first coordinate is equal to the (i+1)st coordinate in Endospec X, and $s_i = 1$ otherwise. The integer

$\sum_{i=1}^{5} s_i 2^{i-1}$ is called the endomorphism type of X and is denoted by Endotype X.

There are 32 possibilities, that is, endotype 0 up to endotype 31. It is known that Endotype 0 describes unretractive graphs, endotype 0 up to 15 describe Sunretractive graphs, endotype 16 describes E-S-unretractive graphs which are not unretractive, endotype 31 describes graphs for which all the 6 sets are different (see [3] and its references).

A mapping f stands for an endomorphism of graph X.The image of X under f is subgraph of X denoted by I_f . $V(I_f) = f(V(x))$ and

$\{f(a), f(b)\} \in E(I_f)$ if and only if $c \in f^{-1}(f(a))$, $d \in f^{-1}(f(b))$ and $\{c,d\} \in E(X)$. $f(C_n)$ is used to represent the image of C_n in the endomorphism f.

2. Endotype Cn and Endspec Cn

The points of graph C_n are denoted by 1,2,...,n according to the inverse time order. The points of graph P_n are denoted by 1,2,...,n from left to right.

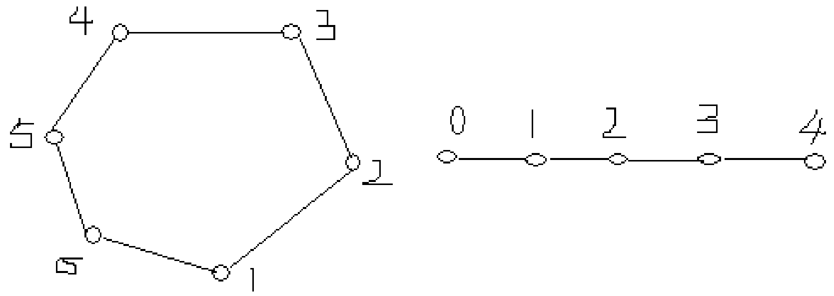

Figure 1 respectively said C_6 and P_4

They are respectively said C_6 and P_4(see figure 1), so the following figure with all its vertex digital according to the above methods said.I.e. $V(C_n)=\{1,2,3,...,n\}$, $\{i,j\}\in E(C_n)$ if and only if $i-j\equiv 1$ or -1(mod n), and denote that $C_n=(1,2,3,...,n)$ at the same time. $V(P_n)=\{1,2,3,...,n\}$, $\{i,j\}\in E(P_n)$ if and only if $|i-j|=1$ And denote that $P_n=(1,2,3,...,n)$ at the same time.Lemma 2.1. Let fbe an endomorphism of graph C_n ,ThenIf n is an even integer, then $f(C_n)=C_n$ or $f(C_n)=P_k$, $k=1,2,3,...,\frac{n}{2}$ If n is an odd integer, then $f(C_n)=C_n$.Proof. (1) The endomorphism transforms a connected graph into a connected graph and the connected subgraph of the cycle isonly itself or a path above it. According to the definition of endomorphism,the length of the longest of the path is only a

half of the length of the cycle.And if the length of the path is only less than a half of the length of cycle, endomorphism exists that can transform the cycle into the path. If a endomorphism f maps odd cycle $(1,2,3,...,n)$ into a section of path $(0^*,1^*,2^*,...,n^*)$,without loss of generality, we can suppose that $f(1)=0^*$. Then we can get $f(1)=1^*$ from $\{1,\ n\}\in E(C_n)$ and $\{1,\ n\}\in E(C_n)$ $f^{-1}(0^*)$, $f^{-1}(1^*)$,..., $f^{-1}(n^*)$ display odd-even alternation,therefore $f^{-1}(1^*)$ is only an even integer,which is a contradiction with $f(n)=1^*$.Consequently, we get that a endomorphism only maps all cycles into odd cycles.Theorem 2.2 :when n is odd, End C_n= Aut (C_n),Endspec C_n=(2n, 2n, 2n, 2n, 2n, 2n) and Endotype C_n=0.

Proof: By lemma 2.2, f (C_n) = C_n, so the endomorphism of which is as well as its automorphisms, it is easy to know that its automorphism forms a dihedral group with the order number 2 , thus the endomorphism spectrum and the endomorphism type are clearly.

Since the endomorphism is relatively simple for a odd cycle,all cycles we discuss in the below are even cycles.For convenience, we use C_{2n} to express an even cycle with length of 2n. The following is mainly the study of the relationship between the six kinds of the endomorphisms of even cycle.

Definition 2.3:If a endomorphism f ofthe cycle C_{2n} satisfies f (i+1) =f(i)+1 at the point of i (the definition of here are all on the Z_{2n}),then we say that f in the i point is a step forward, if f (i+1)+1= f (i), then say that f takes a step back at the point of i. We construct a vector of length 2n corresponding to each endomorphism f ,the ith component of component is definited as : we sign it 1 when f is a step forward,conversely,we sign it 0 when f takes a step back. So each endomorphism corresponding to a vector,but it is not the one -to-one relationship. For example, every endomorphism corresponds to (1,1,...,1) or (-1, -1,... -1). We call the vector whose length is 2n is the vector corresponding to f,and is dedoted by X_{f} .

Definition 2.4: A vector with the sum of all components is zero called a balanced vector,the B (2n) is defined as the set of all balanced vector with length 2n.

When f (C_{2n}) is a road, that is, f $\in$ End(C_{2n})/Aut(C_{2n}),we do not consider the location of the road in the cycle, and only consider the specific way of f mapping in the road, the roads without consideringits location in the cycle are all recorded as:

Pk*= (0*, 1*, 2*,... , k*), k=1,2,3,... N. Road (1,2,3,4) and road (2,3,4,5) are considered as (0*, 1*, 2*, 3*) that some different endomorphisms will become the same homomorphism, according to this classification ,each class has 2n endomorphisms,the endomorphism which according to above classification is denoted as {End (C_{2n}) /Aut (C_{2n})}.

Lemma 2.5:There is a one to one relationship between {End(C_{2n})/Aut(C_{2n})}and B (2n).

Proof: first, we prove that any f $\in$ {End (C_{2n}) /Aut (C_{2n})}, then the corresponding vectors of f are all in B(2n) . It is only needed to prove that X_{f} is a balanced vector. When f $\in$ {End (C_{2n}) /Aut (C_{2n})}, f (C_{2n}) is a road, if the mapping is seen as moving, the starting point of the road starts to move forward ,and then no matter how to move in the middle ,it will go back to the road at last, so the forward and backward steps are the same as the sum of X_{f} is 0, that is, the X_{f} is the balanced vector, $X_{\mathrm{f}} \in$ B (2n).

Finally,we prove that each vector of B（$2n$）corresponds one and only one endomorphism in $\{End(C_{2n})/Aut(C_{2n})\}$.For any $X \in B(2\mathrm{n})$, $X = (x_1, x_2,..., \mathrm{x}_{2\mathrm{n}})$,denoted by $X_i = \Sigma^i_{j=1} x_j$,i.e.the sum of the former i components which denoted by $q = \min_{1 \le i \le 2n} X_i$ .Then if $q = 0$, then $f(1) = 0^*$;if $q < 0$, then $f(1) = |q|^*$,the following is similar to the definition of 2.3,if $x_i = 1$,then $f(i+1) = f(i)+1$;if $x_i =$ -1,then $f(i+1) = f(i)$ -1.Thus f is uniquely defined.Clearly,

according to the corresponding method above,it is impossible to have two vectors make the same homomorphic.

Corollary 2.6 $|End(C_{2n})| = 2n \times (C_{2n} + 2)$

Proof.$|B(2n)| = C^n_{2n}$,select n positions from the 2n positions labeled by 1,and the others labeled by -1,thus,there is having C^n_{2n} .Each element of $B(2n)$ corresponds 2n endomorphisms,and add 4n automorphisms,therefore we can obtain it by computing.

Theorem 2.7 $End(C_{2n}) = HEnd(C_{2n})$

Proof. For any endomorphism f,if $f(C_{2n}) = C_{2n}$,then f is the endomorphism;if $f(C_{2n}) = P_k$,for any two connected points in the path there must exists two points that map f from two connected points in cycle to the path.Therefore f is also the half-strong endomorphism.

Since for the case of cycle 4 is special and it is not easy to deal uniformly,we are listing them separately.

Theorem 2.8 $Endspec \quad C_4 =$（32,32,32,32,32,8）

Proof.We can obtain it by computing according to the six kinds of definition of endomorphism.The following discussions are all carried out for the even cycle whose length is greater or equal to 6. In order to compute the number of locally endomorphism, we need that similar to the definition of the first section in article [4], the preimage set of i^* under f denoted by $[i^*]$.

Definition 2.9 endomorphism f: $V(C_{2n}) \to V(P_k)$ is referred to as a fold (n is divisible by k),if satisfies:

$[0^*] = \{2mk + 1 \in P_k \mid m = 0,1,...\}$,

$[k^*] = \{(2m+1)k + 1 \in P_k \mid m = 0,1,2,...\}$,

$[r^*] = \{2ml + r + 1 \in P_k \mid m = 0,1,...\} \bigcup \{2ml - r + 1 \in P_k \mid m = 1,2,...\}. 0 < r < l$,

now the path is $(0^*,1^*,2^*,...,k^*)$.When the path is $(k^*,...,2^*,1^*,0^*)$,if f satisfies the above as well,then called f is reverse fold.

Lemma 2.10 If $f \in LEnd(\mathrm{C}_{2n})/Aut(\mathrm{C}_{2n})$ and $f(1) = 0^*$,then f is a fold or a reverse fold.

Proof. $f \in LEnd(\mathrm{C}_{2n})/Aut(\mathrm{C}_{2n})$,then the image of C_{2n} under f is path,hence we suppose that $f(C_{2n}) = P_k = (0^*,1^*,2^*,...,k^*)$ .If f is not a fold then either there exists $f(i+2) = f(i) = j^*$, $f(i+1) = (j-1)^*$,at this time since $(j-1)^*, (j-2)^* \in P_k$,the point i+1 is only connected with i or i+2,there is not exists a point a such that $f(a) = (j-2)^*$ and $(a, j+1) \in C_{2n}$ .Therefore,it contradicts to locally

endomorphism.Or exists $f(i+2) = f(i) = j^*$, $f(i+1) = (j+1)^*$,at this time there is not exists a point b such that $f(b) = (j+2)^*$ and $(b, i+1) \in P_k$ .According to the above discussions we know that f must be a fold.

This moment, there is no point b makes $f(b) = (j+2)^*$,and $(b, i+1) \in P_k$, the above discussion is that f must be a folding. When $f(C_{2n}) = P_k = (k^*,...,2^*,1^*,0^*)$, according to the above completely similar to the discussion of the f is a reverse stack.

Theorem 2.11 |LEnd(C2n)|=8n2 ×A(n)+4n,where A(n) is the number of factors of n.

Proof: because f may be a point of {1,2,3,... ,n} map to the beginning of each path, hence the foldings and the reverse stacks totally have 4n which are similar with limiting $f(1) = 0^*$ of lemma , and each factor of n has a folding and a reverse stack of lemma . Hence , The number of the set $LEnd(C_{2n})/Aut(C_{2n})$ is $8n^2 \times A(n)$.Besides, the number of the automorphism is 4n . Hence, $|LEnd(C_{2n})| = 4n \times A(n) + 4n$.

Lemma 2.12 If f is a half-strong endomorphism , then any image of C_{2n} under f can not have two more than the original image .

Proof. If $f(a) = f(b) = f(c) = i^*$, then $(\mathrm{i}^*, (i+1)^*)$ or $(i^*, (i-1)^*)$ on the road, let $(\mathrm{i}^*, (i+1)^*)$ on the road, so need to find a point d such that $f(d) = (i+1)^*$ and (d,a),(d,b),(d,c) are not possible at a same cycle, because every point on the cycle is only connected to two points.

Corollary 2.13 If f is a half-strong endomorphism , then $f(C_{2n}) = C_{2n}$ or $f(C_{2n}) = P_n$.In the first case f is an automorphism , and the second case f is a folding.

Proof . By lemma 2.12, any image of C_{2n} under f can not have two more than the original image .When a point is only one preimage is automorphism; when a point have two preimage, by lemma 2.10, know $f(C_{2n}) = P_n$ is a folding, and other circumstances may not make up only two of the original image.

Theorem 2.14 $QEnd(C_{2n}) = SEnd(C_{2n}) = Aut(C_{2n})$.

Proof . Only prove the $QEnd(C_{2n}) = Aut(C_{2n})$. And by Inference2.13 only rule out the second Circumstance .The following proof $f(C_{2n}) = P_n$ is not possible.

If the folding of f makes $f(C_{2n}) = P_n$, by folding the definition know that f(2)=f(2n), where n = 3 , hence ,it is impossible that there is a point a making (a,2) and (a,2n) are in the cycle, which contradicts to quasi-strong endomorphism f .

Theorem 2.15 For n≥3, Endspec C_{2n}

$$=(2n\times(C_{2n}^{n}+2),\ 2n\times(C_{2n}^{n}+2),\ 8n^2\times A(n)+4n,4n,4n,4n)$$

where A(n) is the function of the number of factors of n.

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References

[1] C. Godsil, G. Royle, Algebra Graph Theory, Springer-Verlag, New York, 2000.
[2] J.M. Howie, Fundamentals of Semigroup Theory, Clarendon Press, Oxford, 1995.
[3] M. B ̈ottcher, U. Knauer, Endomorphism spectra of graphs, Discrete Math. 109 (1992) 45–57.
[4] Sr. Arworn, U. Knauer, S. Leeratanavalee，Locally strong endomorphisms of paths
[5] Hailong Hou, Yanfeng Luo, Zhimi Cheng ,The endomorphism monoid of Pn
[6] W. Li, J. Chen, Endomorphism-regularity of split graphs, European J. Combin. 22 (2001) 207–216.
[7] W. Li, A regular endomorphism of a graph and its inverse, Mathematik 41 (1994) 189–198.
[8] L. Marki, Problems raised at the problem session of the Colloquium on Semigroups in Szeged, August 1987,Semigroup Forum 37 (1988) 367–373.
[9] P. Petrich, Norman R. Reilly, Completely Regular Semigroups, John Wiley & Sons, Inc., New York, 1999.
[10] E. Wilkeit, Graphs with regular endomorphism monoid, Arch. Math. 66 (1996) 344–352.