Optimization consensus and event-based control

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Abstract

In this paper, we study the distributed optimization problem with a common state set constraint. And the objective function is the summation of local objective function, where each component is only known to particular node. In order to further reduce the number of times about update in the controller and the number of exchanges, the state in the controller is further sampled based on previous work. It is shown that the constrained optimal consensus can be achieved under event-based distributed control with fixed undirected graph.

Keywords

Event-based control; optimization; consensus; multi-agent optimization.

1. Introduction

From the eighties of last century, [1] discusses the situation of global target optimization functions that each individual knows. Due to the huge practical value of distributed optimization, more and more scholars have joined the research on related issues[2][3][21][22].[2][11] first discussed the more special but also more practical goal optimization function on the basis of the above work[1]. The objective optimization function is composed of a set of local objective functions. Each local objective function is only for the individual. Obviously, this form of objective function has more obvious practical engineering significance. In the actual project, people do not want the state to reach a certain intolerant range, which is mathematically represented by state constraints at this time.[4] further proposes a distributed optimization algorithm with state constraints. Over the past few years, many scholars have also discussed the abstract constraint range or the concrete equality constraint inequality constraint for distributed optimization [5] [6]. In the study of distributed optimization problem, whether the objective function can achieve global optimization extremely depends on joint design of information connection structure and the control decision made by the individual in time and space. Because of the special structure of the optimization function (the weighted average sum about all local functions), the global objective optimization is dependent on the balance analysis of the graph under the control algorithm of states convexity. The article[7][8]takes the push-sum[9][10] algorithm, which make Implementation of global goals strengthen the ability to resist interference from the graph, and the achievement of global goals no longer depends on the balance of graphs.[2] [4] [8]shows that it is extremely important for the distributed system individual to collaborate to complete the global target task and to design a sufficient distributed control algorithm.

As the consistent algorithm based on domain information is proposed in the article [12], the multi -agent system has received more and more attention in recent years. Many scholars have developed a large number of practical engineering value of the control algorithm under the idea of above control decision. Such as an event-based control algorithm[13][14][15]that can reduce the controller update calculation and reduce the number of individual exchanges. When this problem is transformed into a mathematical model, its analytical nature will belong to the analysis of the stability of the hybrid system analysis. This paper will focus on the application of this kind of engineering algorithm based on event control in distributed optimization.

Many important distributed optimization works is based on discrete models. However, with the depth of the study about distributed optimization problems, the discussion of continuous model about it began to be introduced, because of obvious geometric significance under the continuous model. The system of the continuous mode would discuss whether the trajectory of the solution state under the differential state equation converges to the target region in the state space. In the article [16], under the time varying joint strongly directed graph, a control input compose of the state-averaged term and the local optimal term is designed about the continuous system. Through the joint action above two terms, the state would achieve consistent and gradually converges to the global optimal set (the intersection of all local optimal sets). In this paper [17],

a control algorithm composed of uniform average term, constrained projection term and local objective function gradient optimization term under continuous system is proposed under the time - varying joint connected undirected graph. Under these three conditions, states of system reach consensus and convergence the global optimal set. When the system is considered of the continuous model, how to reduce update and calculation in the controller and reduce the number of exchanges is inspired by the event control based on continuous model naturally. In the article [19], the author designs a novel driving function and control algorithm to achieve event-based distributed optimization. In addition to the above work to consider event-based distributed optimization problems, there are some works to discuss this issue[18][20][23][24][25],which no longer be elaborated one by one.

To Summary, the state of the controller would do a further sampling processing based on[19], in order to further reduce the controller update. When the latest sampling state and the true state of the error is equal to the dynamic boundary we design, the controller updates. We use the way of analysis proposed in[19], which is dependent on design skills based on property of step. In this paper, we prove that the state converges to the set of constraints and the consistent set, and the states achieve the global optimization would be proved, finally.

2. Mathematical Preliminaries

2.1 Graph Notation

The communication topology among these agents is represented by $\mathcal{G}=(\mathcal{V},\varepsilon)$, $\mathcal{V}=\left(1,2,\ldots,n\right)$ is the set of nodes: ε is the set of edges which are represented by a pair of node indices $\left(i,j\right)$. We shall write $\left(i,j\right)\in\varepsilon$ if node I can send its information to node j.

The matrix $A=\left[a_{i,j}\right]\in\mathbb{R}^{n\times n}$ is called the weighted adjacency matrix associated with $\mathcal{G}$; $a_{ij}>0$ if $\left(i,j\right)\in\varepsilon$ and $a_{ij}=0$,otherwise. A graph has a spanning tree if there exists a node such that there is at least one directed path from the node to each of the other nodes. A digraph $\mathcal{G}$ is "balanced" if the in-degree $d_{out}(i)\triangleq\sum_{j\in\mathcal{V}}a_{j,i}$ and the out-degree $d_{out}(i)\triangleq\sum_{j\in\mathcal{V}}a_{j,i}$ are equal for $i\in\mathcal{V}$. We denote $d=\max_i d_{in}(i)$ the degree of $\mathcal{G}$ and $D=diag\{D_{in}(1),\ldots,D_{in}(n)\}$ the degree matrix of $\mathcal{G}$. The Laplacian matrix $\mathcal{L}$ is defined as $\mathcal{L}=D-A$ .Some properties of $\mathcal{L}$ are summarized in the following lemma:

Lemma 2.1[29] If a graph $\mathcal{G}$ is balanced ,then there exists a standard orthogonal matrix $\Phi=[\frac{1_n}{\sqrt{n}},\phi]$ with $\phi\in\mathbb{R}^{n\times n-1}$ such $\Phi'\mathcal{L}\Phi=diag(0,\bar{\mathcal{L}})$ where $\bar{\mathcal{L}}\in\mathbb{R}^{(n-1)\times(n-1)}$ If $\mathcal{G}$ further contains a spanning tree, then all eigenvalues of $\bar{\mathcal{L}}$ have positive real parts.

2.2 Convex Analysis

A set $\mathcal{C}\in\mathbb{R}^m$ is convex, if $\forall x,y\in\mathcal{C}$ and $\theta\in(0,1)$,$\theta x+(1-\theta)y\in\mathcal{C}$ A function $f(\cdot):\mathbb{R}^m\rightarrow\mathbb{R}$ is convex if $\forall x,y\in\mathcal{C}$ and $\theta\in(0,1)$, $f(\theta x+(1-\theta)y)\leq\theta f(x)+(1-\theta)f(y)$.

For the convex is non-differentiable, a sub-differential is introduced. Consider a function $f:\mathbb{R}^m \to \mathbb{R}$ the set: $\partial f(x)=\{s_f(x)\in\mathbb{R}^m \,|\, f(y)-f(x)\ge\langle s_f(x),y-x\rangle,\forall x\in\mathbb{R}^m\}$ is the "sub-differential of f at point x ".The element $s_f(x)$ is called "the subgradient of f at x ".A function f is "δ-Strongly convex" with respect to the Euclidean norm, if for all x , y in relative interior of domain of f ,and $\theta\in(0,1)$: $f(\theta x+(1-\theta)y)\le\theta f(x)+(1-\theta)f(y)-\frac{1}{2}\delta\theta(1-\theta)|x-y|^2$.

One property of a δ-strongly convex function f is $f(y)-f(x)\ge\langle s_f(x),y-x\rangle+\frac{1}{2}|x-y|^2$.

For a closed convex set $\mathcal{C}\in\mathbb{R}^m$. $P_{\mathcal{C}}(x):\mathbb{R}^m\to\mathcal{C}$ denote a projection function satisfying:

$$|x-P_{\mathcal{C}}(x)|=\inf_{v\in\mathcal{C}}|x-v|\triangleq|x|_{\mathcal{C}},\forall x\in\mathbb{R}^m$$

A non-expansive property of $P_{\mathcal{C}}(\cdot)$ is given below:

$$|P_{\mathcal{C}}(x)-P_{\mathcal{C}}(y)|\le|x-y|,\forall x,y\in\mathbb{R}^m .$$

The following lemma will be used in subsequent parts of this work:

Lemma 2.2[17] Given a closed convex set $\mathcal{C}\in\mathbb{R}^m$ and $x,y\in\mathbb{R}^m$,we have

$$\langle x-P_{\mathcal{C}}(x),y-x\rangle\le|x|_{\mathcal{C}}(|y|_{\mathcal{C}}-|x|_{\mathcal{C}})$$

Consider the following ordinary differential equation:

$$\dot{x}(t)=f(x(t),t),x(t_0)=x_0\in\mathbb{R}^m \tag{1}$$

Where $f(\cdot,\cdot):\mathbb{R}^m\times\mathbb{R}\to\mathbb{R}$ is not necessarily continuous. Classical solution may not exist due to the discontinuity of f .Now we consider the solution of (1) in term of the following differential inclusion:

$$\dot{x}(t)\in\mathcal{F}[f](x,t)\triangleq\bigcap_{\delta>0}\bigcap_{\mu(s)=0}co\{f(B(x,\delta)\setminus S,t)\}$$

Lemma2.3[27] the function $f(t,x)$ is upper semi-continuous and is compact convex for any $(x_0,t_0)\in\mathcal{D}$.

Let $(x,t)\in\mathcal{D}$ for all $t\in[a,b]$. $x(t)$ is absolutely continuous on $[a,b]$ and $\dot{x}(t)\in f(x(t),t),x(t_0)=x_0$ holds almost everywhere.

2.3 Non-smooth Analysis and Discontinuous Differential Equation

Lemma 2.4[28] Let $V_i(t,x):\mathbb{R}\times\mathbb{R}^m\to\mathbb{R}$, $i=1,2,\dots,n$ be continuously differentiable and $V(t,x)=\max_{i=1,2,\dots,n}V_i(t,x)$, if $\mathcal{I}(t)=i$: $V(t,x)=V_i(t,x)$ denotes the of indices where the maximum $V(t,x)$ is reacted at t ,then $D^+V(t,x)=\max_{i\in\mathcal{I}}V_i(t,x)$.

3. Problem Formulation And Algorithm Design

3.1 Problem Formulation

In this section, we define the considered optimal consensus problem. Consider the following continuous-time multi-agent system

$$\dot{x}_i(t)=u_i(t),i\in\mathcal{V}\triangleq\{1,\dots,n\} \tag{2}$$

Where each $x_i(t)\in\mathbb{R}^m$ is the state of agent i . The communication topology can be represented by a directed graph $\mathcal{G}$.The connectivity of the communication graph is assumed as follows:

assumption 3.1 The Graph is balanced and contains a spanning tree.

Each agent i is assigned with a cost function $f_i : \mathbb{R}^m \to \mathbb{R}$, $i \in \mathcal{V}$. The local cost is represented by $f_i(x_i)$ which is only known to agent i. The objective is that as $t \to \infty$ the states of all agents approach a global minimum point of the function $F(x) = \sum_{i\in\mathcal{R}} f_i(x)$ within constrain set $\mathcal{X} \subset \mathbb{R}^m$ by designing $u_i(t)$ based on local information, i.e.

$$\limsup_{t\to\infty} \left| x_i(t) \right|_{\mathcal{X}^*} = 0, \forall i \in \mathcal{V} . \tag{3}$$

Where $\mathcal{X}^* = \{v \in \mathcal{X} : \arg\min F(v)\}$ is the optimal set. To facilitate our subsequent analysis, the following assumptions are imposed:

assumption 3.2 The constraint set X is a closed convex set.

assumption 3.3 Each f_i is δ-strongly convex.

As a consequence of Assumption 2 and 3, the global optimal set $\mathcal{X}$ is a singleton, which is denoted as x^*.

3.2 Algorithm Design

We propose the following algorithm:

$$u_i(t) = -\alpha(t)s_i(x_i(t_k^i)) + \beta(P_{\mathcal{X}}(x_i(t_k^i)) - x_i(t_k^i)) + \gamma\sum_{j\in\mathcal{V}} a_{i,j}((t_k^j) - (t_k^i)), t \in [t_k^i, t_{k+1}^i). \tag{4}$$

Where $s_i(x_i(t))$ is a subgradient of f_i at x_i and $\alpha(t)$ is a positive gain function of time t. β and γ are positive gainon the projection and averaging terms,respectively. $\{(t_k^i), k = 1, 2\cdots\}$ are triggered time instants for agent i to broadcast its state information.Boradcasting only happens when $T_i(t) \le 0$ with $T_i(t)$ the triggered function of agent i to be designed. We use t_k^i to indicate the last triggered time of agent i at time instant t ,i.e. $t_k^i = \max_\tau \{\tau \le t : T_i(\tau) \le 0\}$ .The corresponding Laplacian matrix is denoted as $\mathcal{L}$ .Without loss of generality, we assume that $d = \max_i d_{in} \le 1$.Otherwise,we can rescale γ and $a_{i,j}$ to make this hold.

For each agent i in system,we choose trigger function of the form:

$$T_i(t) = \theta(t)\alpha(t) - |e_i(t)|_\infty, e_i(t) = x_i(t_k^i) - x_i(t) . \tag{5}$$

where $\theta(t)$ is a uniformly bounded positive continuous function. Note that agent i broadcast its state information only when $|e_i(t)|_\infty$ goes beyond the time-varying threshold $\theta(t)\alpha(t)$. We impose the following assumption: the function $\theta(t)$ is differential and $\lim_{t\to\infty}\theta(t) = 0$.

$\alpha(t)$ is continuously differentiable and satisfies the following conditions:

$$\int_0^{+\infty} \alpha(t), \ \lim_{t\to\infty} \alpha(t) = 0, \alpha(t) = o(\alpha(t)). \tag{6}$$

3.3 Convergence Analysis Results

In this section,we should analyze the convergence of the proposed algorithm.Note that due the subgradient being discontinuous and the fact that the neighbours' information may jump at triggering time instants, the following inclusion should be considered:

$$u_i(t) \in \mathcal{F}[-\alpha(t)s_i(x_i(t_k^i)) + \beta(P_{\mathcal{X}}(x_i(t_k^i)) - x_i(t_k^i)) + \gamma\sum_{j\in\mathcal{V}} a_{i,j}((t_k^j) - (t_k^i))] \tag{7}$$

Substituting into and considering yield that:

$$\dot{x}_i(t) \in \mathcal{F}[-\alpha(t)s_i(x_i(t_k^i)) + \beta(P_{\mathcal{X}}(x_i(t_k^i)) - x_i(t_k^i)) + \gamma\sum_{j\in\mathcal{V}} a_{i,j}((t_k^j) - (t_k^i))] \tag{8}$$

We will regard the states $x_i(t)$ as the Filippov solutions to the above differential inclusions.By assumption,for any ε ,there exists t_ε such that:

$$\theta(t)\in(0,\varepsilon),\dot{\theta}(t)\in(-\varepsilon,\varepsilon),|\dot{\alpha}(t)\alpha^{-1}(t)|\le\varepsilon,\forall t\ge t_\varepsilon. \tag{9}$$

Before giving the convergence analysis of proposed algorithm,The following preliminary results are introduced. Proposition 4.1 establishes boundedness of the states and Proposition 4.2 shows asymptotic convergence of states of agents to a agreement which lies in constraint set $\mathcal{X}$.

Proposition 4.1. We apply (4) to multi-agent system(2),and the communication among the agents is triggered based on trigger function (5). We can obtain The states of all agents are uniformed bounded.

Proof: Consider $d_i(t)=max_{i\in\mathcal{V}}d_i(t)$ with $d_i(t)=|x_i(t)|^2, y\in\mathcal{X}$ and $\mathcal{I}(t)=\{i\in\mathcal{V}:q_i(t)=q(t)\}$. Thus $\mathcal{I}$ represents the set of indices of agents whose states are farthest from y . According to Lemma 1 ,for all $i\in\mathcal{I}$ and $j\in\mathcal{V}$.

According to Assumption, f_i is strongly convex and there exists $r_i>0$ such that $f_i(x)>f_i(y)$ if $|x-y|\ge r_i$, in light of $\langle x-y,s_i(x)\rangle\ \ge\ f_i(x)-f_i(y)+\frac{\delta}{2}|x-y|^2$, we can find $\langle x-y,\ s_i(x)\rangle\ >\ \frac{\delta}{2}|x-y|^2$. Then r can be selected as $r=\max_{i\in\mathcal{V}}r_i$.

$$\dot{d}(t)=2\langle x_i(t)-y,-\alpha(t)s_i(x_i(t_k^i))+\beta(P_{\mathcal{X}}(x_i(t_k^i))-x_i(t_k^i))+\gamma\sum_{j\in\mathcal{V}}a_{i,j}((t_k^j)-(t_k^i))\rangle \tag{10}$$

Then from (10), if $d(t)\ge r$, according to lemma 2 and lemma 3, we have

$$\begin{aligned}\dot{d}(t)=2\langle x_i(t)-y,&-\alpha(t)(s_i(x_i(t_k^i))-s_i(x_i(t)))-\alpha(t)s_i(x_i(t))\\&+\beta(P_{\mathcal{X}}(x_i(t_k^i))-P_{\mathcal{X}}(x_i(t))+P_{\mathcal{X}}(x_i(t))-x_i(t)+x_i(t)-x_i(t_k^i))\\&+\gamma\sum_{j\in\mathcal{V}}a_{i,j}(x_j(t)-x_i(t)+e_j(t)-e_i(t))\end{aligned} \tag{11}$$

Because subgradient $s_i(x)$ is bounded,so there exists a positive constant L , $|s_i(x_i(t_k^i))-s_i(x_i(t))|$, $\le L|x_i(t_k^i)-x_i(t)|$, and in light of lemma,

$$|P_{\mathcal{X}}(x_i(t_k^i))-P_{\mathcal{X}}(x_i(t))|\le|x_i(t_k^i)-x_i(t)|\le\sqrt{m}|e_i(t)|_\infty\le\sqrt{m}\theta(t)\alpha(t)$$

$$\langle x_i(t)-y,P_{\mathcal{X}}(x_i(t))-x_i(t)\rangle\le-|x_i|_{\mathcal{X}}^2$$

then we get

$$\begin{aligned}\dot{d}(t)&\le-\alpha(t)d(t)+2\sqrt{d(t)}\alpha(t)L|e_i(t)|\\&\quad+4\beta\sqrt{m}\theta(t)\alpha(t)\sqrt{d(t)}+\langle x_i(t)-y,\gamma\sum_{j\in\mathcal{V}}a_{i,j}(e_j(t)-e_i(t))\rangle\\&\le-\delta\alpha(t)d(t)+2\alpha(t)L\theta(t)\sqrt{md(t)}\end{aligned} \tag{12}$$

and then, we show that:

$$\limsup_{t\to\infty}d(t)\le\max\{d(0),\frac{2\alpha(t_k^i)L\varepsilon\sqrt{m}+4(\beta+\gamma)\varepsilon\alpha(t)\sqrt{m}}{\delta\alpha(t_k^i)}\} \tag{13}$$

where ε , t_ε are defined in (9).

Suppose $d(t)\ge\max\{\bar{d},(\frac{2\alpha(t_k^i)L\varepsilon\sqrt{m}+4(\beta+\gamma)\varepsilon\alpha(t)\sqrt{m}}{\delta\alpha(t_k^i)}+\varepsilon)^2\}$ for $t\in(t_\varepsilon,\infty)$, where $\bar{d}$ is a positive constant.According to (11), we have

$\dot{d}(t) \le 0$

which show that $d(t)$ would asymptotically convergence to 0. The arbitrariness of ε. ○

From Proposition 4.1, we know that there exists such that $\max_{i\in\mathcal{V}} \sup_{t\in\mathbb{R}^+} | x_i(t) | \le q$. Then, we can define $\bar{s} = \max_i \in \mathcal{V}\{| s_i(v) |_\infty : s_i \in \partial f_i, | v | \le q\}$ and note that $sup_{t\in\mathbb{R}^+} \le \bar{s} < +\infty$.

Proposition 4.2. Consider multi-agent system (2) with u_i given in (4). Under Assumption 3.1-3.3, then $\forall i, j$

$$\limsup_{t\to\infty} \frac{| x_i(t) |_{\mathcal{X}}}{\alpha(t)} \le C_1, \quad (14)$$

$$\limsup_{t\to\infty} \frac{| x_i(t) - x_j(t) |}{\alpha(t)} \le C_2, \quad (15)$$

Where $C_1 = \frac{\bar{\alpha}\bar{s}}{\beta}, C_2 = \frac{2\sqrt{n}\bar{s}(1+\bar{\alpha})}{\gamma\lambda_2}$.

Proof: To measure the distance of the states to $\mathcal{X}$, we introduce $h(t) = \max_{i\in\mathcal{V}} | x_i(t) |^2_{\mathcal{X}}$ and $\mathcal{I}(t) = \{i \in \mathcal{V} : | x_i(t) |^2_{\mathcal{X}} = h(t)\}$, where $\mathcal{I}(t)$ is the set of indices of agent whose states are farthest from the constraint set $\mathcal{X}$ at time t. Convergence to constraint set $\mathcal{X}$ is equivalent to the requirement $\lim_{t\to\infty} h(t) = 0$. According to (4) and lemma 2.2 , we have

$$\begin{aligned} \dot{h}(t) &= 2\langle x_i(t) - P_{\mathcal{X}}(x_i(t\ )), -\alpha(t)s_i(x_i(t_k^i)) + \beta(P_{\mathcal{X}}(x_i(t_k^i)) - x_i(t_k^i)) + \gamma\sum_{j\in\mathcal{V}} a_{i,j}((t_k^j) - (t_k^i))\rangle \\ &= 2\langle x_i(t) - P_{\mathcal{X}}(x_i(t\ )), -\alpha(t)s_i(x_i(t_k^i)) + \gamma\sum_{j\in\mathcal{V}} a_{i,j}(x_j(t) - x_i(t) + e_j(t) - e_i(t))\rangle \\ &\quad + \beta(P_{\mathcal{X}}(x_i(t_k^i)) - P_{\mathcal{X}}(x_i(t)) + P_{\mathcal{X}}(x_i(t)) - x_i(t) + x_i(t) - x_i(t_k^i)) \end{aligned} \quad (16)$$

According to non-expansion of projection and lemma 2.2 we can obtain:

$$| P_{\mathcal{X}}(x_i(t_k^i)) - P_{\mathcal{X}}(x_i(t)) | \le | x_i(t_k^i) - x_i(t) | \le \sqrt{m} | e_i(t) |_\infty \le \sqrt{m}\theta(t)\alpha(t)$$

$$\dot{h}(t) \le 2\langle x_i(t) - P_{\mathcal{X}}(x_i(t\ )), -\alpha(t)s_i(x_i(t_k^i))\rangle - 2\beta h(t) + 4\beta\sqrt{m}\theta(t)\alpha(t)\sqrt{h(t)} + 4\gamma\sqrt{m}\theta(t)\alpha(t)\sqrt{h(t)} \quad (17)$$

Due to the property of $\theta(t)$ and $\alpha(t)$, there exists $\theta(t)\alpha(t) \le \varepsilon$ when $t \in (t_\varepsilon, \infty)$

$$\dot{h}(t) \le -2\beta h(t) + 2\alpha(t)\bar{s}\sqrt{h(t)} + 4\beta\sqrt{m}\varepsilon\alpha(t)\sqrt{h(t)} + 4\gamma\sqrt{m}\varepsilon\alpha(t)\sqrt{h(t)} \quad (18)$$

We make $\hat{h}(t) \triangleq \frac{h(t)}{\alpha^2(t)}$,

then.

$$\begin{aligned} \dot{\hat{h}}(t) &= \frac{\dot{h}(t)}{\alpha^2(t)} - 2\frac{\dot{\alpha}(t)}{\alpha^3(t)} h(t) \\ &\le (-2\beta - 2\frac{\dot{\alpha}(t)}{\alpha(t)})\hat{h}(t) + (4(\beta+\gamma)\sqrt{m}\theta(t) + 2\bar{s})\sqrt{\hat{h}(t)} \end{aligned} \quad (19)$$

Since the above inequality holds for any $\varepsilon \in \mathbb{R}^+$, we consider $\varepsilon \in (0, \beta)$.We consider the following ordinary differential equation

$$\dot{\hat{h}}(t) = (-2\beta - 2\varepsilon)\hat{h}(t) + (4(\beta+\gamma)\sqrt{m}\varepsilon + 2\bar{\alpha}\bar{s})\sqrt{\hat{h}(t)}$$

with initial equation $\hat{h}(t_\varepsilon)$,we can obtain

$$\hat{h}(t)=[\frac{2\sqrt{m}(\beta+\gamma)\varepsilon+bar\alpha\overline{s}}{\beta-\varepsilon}+e^{-(\beta-\varepsilon(t-t_\varepsilon))}\times(\sqrt{\hat{h}(t_\varepsilon)}-\frac{2\sqrt{m}(\beta+\gamma)\varepsilon+bar\alpha\overline{s}}{\beta-\varepsilon})]^2 \quad (20)$$

and that $\lim_{t\to\infty}\hat{h}(t)=[\frac{2\sqrt{m}(\beta+\gamma)\varepsilon+bar\alpha\overline{s}}{\beta}]^2$, which implies (14) by the arbitrariness of ε.

Next we would prove (15), for Laplacian matrix $\mathcal{L}$ of communication graph $\mathcal{G}$, there exists a standard orthogonal matrix $\Psi=[\frac{\mathbf{1}_n}{\sqrt{N}},\psi]$ such that $\Psi'\mathcal{L}\Psi=diag(0,\mathcal{L}_1)$.

Define $\hat{X}(t)\triangleq(\psi\otimes I_m)X(t)$, $X(t)=[x_1'\cdots x_n']$. Then from (8) we have:

$$\dot{X}(t)=-\gamma(\mathcal{L}\otimes I_m)X(t)+v(t)$$

And

$$\dot{\hat{X}}(t)=-\gamma(\mathcal{L}_1\otimes I_m)\hat{X}(t)+(\psi\otimes I_m)v(t)$$

$$\in-\gamma(\mathcal{L}_1\otimes I_m)\hat{X}(t)+(\psi\otimes I_m)\mathcal{F}[v(t)] \quad (21)$$

Where

$$v(t)\triangleq-\alpha(t)[s_1'(x_1)\cdots s_n'(x_1)]'+\beta[P_{\mathcal{X}}'(x_1(t_k^i))-x_1'(t_k^i)\cdots P_{\mathcal{X}}'(x_n(t_k^i))-x_n'(t_k^i)]'+\gamma(\mathcal{L}\otimes I_m)[e_1'\cdots e_n']$$

due to the conclusion (14), such $|P_{\mathcal{X}}(x_i(t_k^i))-x_i(t_k^i)|\le(C_1+\varepsilon_1)\alpha(t)$,

$$|v(t)|\le\alpha(t)\sqrt{n\overline{s}}+\alpha(t)\beta\sqrt{n}(C_1+\varepsilon_1)+\alpha(t)\gamma\sqrt{mn}\lambda_n^m\varepsilon.$$

Define $X_\alpha(t)\triangleq\frac{\hat{X}(t)}{\alpha(t)}$ By defining $W(t)=\langle X_\alpha(t),X_\alpha(t)\rangle$,

we have $\dot{X}_\alpha\in-(\frac{\dot{\alpha}(t)}{\alpha(t)}+\gamma\mathcal{L}_1\otimes I_m)X_\alpha(t)+\frac{1}{\alpha(t)}(\psi)'\otimes I_m\mathcal{F}[v(t)]$

$$\dot{W}(t)=2\langle X_\alpha,\dot{X}_\alpha\rangle\in2\langle X_\alpha(t),-(\frac{\dot{\alpha}(t)}{\alpha(t)}+\gamma\mathcal{L}_1\otimes I_m)X_\alpha(t)+\frac{1}{\alpha(t)}(\psi)'\otimes I_m\mathcal{F}[v(t)]\rangle$$

$$\le-2(\gamma\lambda_2+\frac{\dot{\alpha}(t)}{\alpha(t)})W(t)+2\sqrt{n}(\overline{s}+\beta(C_1+\varepsilon_1)+\gamma\lambda_n^m\sqrt{m}(\hat{\theta}(t)+\varepsilon))\sqrt{W(t)}$$

In light of the proof of (14),we can obtain

$$\limsup_{t\to\infty}W(t)\le n(\frac{\overline{s}+\beta C_1+\gamma\lambda_n^m\sqrt{m}\hat{\theta}}{\gamma\lambda_2})^2, \quad (22)$$

where λ_2,λ_n is the smallest and biggest eigenvalue of $\frac{\mathcal{L}_1+\mathcal{L}_1'}{2}$,respectively. Let $I_{i,j}\in\mathbb{R}^n$ be the vector with i-th and j-th components being 1 and -1 respectively and the other components being zero. We find that $x_i(t)-x_j(t)=(I_i',j\psi\otimes I_m)\hat{X}(t)$. $|x_i(t)-x_j(t)|\le\sqrt{2}|X_\alpha(t)|\le\sqrt{2W(t)}\alpha(t)$, $,\forall t\ge t_{\varepsilon_1}$. so we obtain $\limsup_{t\to\infty}\frac{|x_i(t)-x_j(t)|}{\alpha(t)}\le\frac{2\sqrt{n\overline{s}}(1+\overline{\alpha})}{\gamma\lambda_2}$.

From Proposition 4.2, we know that there exists $t_{\varepsilon_1}\in[t_\varepsilon,\infty)$ such that $\forall i,j\in\mathcal{V},\forall t\ge t_{\varepsilon_1}$,where ε and t_ε are defined in (9). We will next show that all the states of agents converge to a bounded region of optimum point.

Theorem 1. Consider system (9) with $u_i(t)$ given in (4). Under the same condition as in Proposition 4.2, (3) is achieved.

Proof: Denote $V_i(t)=|x_i(t)-x^*|^2$, and $V(t)=\sum_{i\in\mathcal{V}}\mathcal{V}_i(t)$.

Then we have

$$\dot{V}_i(t)=2\langle x_i(t)-x^*,-\alpha(t)s_i(x_i(t_k^i))+\beta(P_{\mathcal{X}}(x_i(t_k^i))-x_i(t_k^i)) +\gamma\sum_{j\in\mathcal{V}}a_{i,j}((t_k^j)-(t_k^i))\rangle$$

$$=2\langle x_i(t)-x^*,-\alpha(t)s_i(x_i(t_k^i))+\beta(P_{\mathcal{X}}(x_i(t_k^i))-P_{\mathcal{X}}(x_i(t))+P_{\mathcal{X}}(x_i(t))-x_i(t)+x_i(t)-x_i(t_k^i)) +\gamma\sum_{j\in\mathcal{V}}a_{i,j}((x_i(t_k^j)-x_i(t_k^i))$$

$$\dot{V}_i(t)=2\langle x_i(t)-x^*,-\alpha(t)s_i(x_i(t_k^i))+\beta(P_{\mathcal{X}}(x_i(t_k^i))-P_{\mathcal{X}}(x_i(t))+P_{\mathcal{X}}(x_i(t))-x_i(t)+x_i(t)-x_i(t_k^i)) +\gamma\sum_{j\in\mathcal{V}}a_{i,j}(x_j(t)-x_i(t)+e_j(t)-e_i(t))\rangle$$

Recalling lemma 2.1 and $a_{i,j}=a_{j,i}$, we have

$$2\sum_i\sum_j a_{i,j}\langle x_i(t)-x^*,x_j-x_i\rangle$$

$$\le 2\sum_i\sum_j a_{i,j}\sqrt{V_i(t)}(\sqrt{V_j(t)}-\sqrt{V_i(t)})$$

$$=-\sum_i\sum_j a_{i,j}(\sqrt{V_j(t)}-\sqrt{V_i(t)})^2\le 0$$

So we can obtain

$$\dot{V}_i(t)\le 2\langle x_i(t)-x^*,-\alpha(t)s_i(x_i(t_k^i))+\beta(P_{\mathcal{X}}(x_i(t_k^i))-P_{\mathcal{X}}(x_i(t))+P_{\mathcal{X}}(x_i(t))-x_i(t)+x_i(t)-x_i(t_k^i)) +\gamma\sum_{j\in\mathcal{V}}a_{i,j}(e_j(t)-e_i(t))$$

$$|P_{\mathcal{X}}(x_i(t_k^i))-P_{\mathcal{X}}(x_i(t))|\le|x_i(t_k^i)-x_i(t)|\le\sqrt{m}|e_i(t)|_\infty\le\sqrt{m}\theta(t)\alpha(t)$$

$$\langle x_i(t)-x^*,P_{\mathcal{X}}(x_i(t))-x_i(t)\rangle\le-|x_i|_{\mathcal{X}}^2$$

So we can obtain:

$$\dot{V}_i(t)\le 2\langle x_i(t)-x^*,-\alpha(t)s_i(x_i(t_k^i))\rangle+2\gamma\sqrt{V_i(t)}\sum_{j\in\mathcal{V}}a_{i,j}\cdot 2\max_{i\in\mathcal{V}}\{|e_i(t)|\}+4\beta\sqrt{mV_i(t)}\theta(t)\alpha(t)$$

$$\le 2\langle x_i(t)-x^*,-\alpha(t)s_i(x_i(t_k^i))\rangle+4\gamma\sqrt{mV_i(t)}\theta(t)\alpha(t)+4\beta\sqrt{mV_i(t)}\theta(t)\alpha(t)$$

Next let we deal with the $\langle x_i(t)-x^*,-\alpha(t)s_i(x_i(t_k^i))\rangle$,

$$\dot{V}_i(t)2\langle x_i(t)-x^*,-\alpha(t)s_i(x_i(t_k^i))\rangle+4\gamma\sqrt{mV_i(t)}\theta(t)\alpha(t)+4\beta\sqrt{mV_i(t)}\theta(t)\alpha(t)$$

$$\le 2\langle x_i(t)-x^*,-\alpha(t)s_i(x_i(t))-\alpha(t)(s_i(x_i(t_k^i))-s_i(x_i(t))\rangle\backslash+4(\beta+\gamma)\sqrt{mV_i(t)}\theta(t)\alpha(t)$$

$$\le 2\alpha(t)(f_i(x^*)-f_i(x_i(t)))+2\alpha(t)L|e_i(t)|\sqrt{V_i(t)}+4(\beta+\gamma)\sqrt{mV_i(t)}\theta(t)\alpha(t)$$

Where we make use of the property about sub-gradient and $|s_i(x_i(t_k^i))-s_i(x_i(t))|\le L|x_i(t_k^i)-x_i(t)|$, $=L|e_i(t)|$, due to boundedness of $x_i(t)$ and $s_i(\cdot)$.

Now, we think the average state $x(t)\triangleq\frac{1}{n}\sum_{i\in\mathcal{V}}(x_i(t))$.By recall (14)(15),when $t\in[t_{\varepsilon_1},\infty)$ we further implies the following inequality:

$$f_i(x^*)-f_i(x_i(t))=f_i(x^*)-f_i(P_{\mathcal{X}}(\overline{x})+f_i(P_{\mathcal{X}}(\overline{x})-f_i(\overline{x})+f_i(\overline{x})-f_i(x)\le f_i(x^*)-f_i(P_{\mathcal{X}}(\overline{x}))+\varepsilon_1$$

Where $\varepsilon_1=\overline{s}(C_1+C_2+2\varepsilon)\alpha(t)$

On both sides of inequality sum at the same time, and we have

$$\dot{V_i}(t)\le 2\alpha(t)(f_i(x^*)-f_i(P_{\mathcal{X}}(\overline{x}))+\varepsilon_1)+2\alpha(t)\theta(t)L\sqrt{mV_i(t)}\alpha(t)+\alpha(t)4\theta(t)(\beta+\gamma)\sqrt{mV_i(t)} \quad (23)$$

Since F is δ-strongly convex, we have

$$F(P_{\mathcal{X}}(\overline{x}))-F(x^*)\ge\frac{\delta}{2}|P_{\mathcal{X}}(\overline{x})-x^*|^2. \ (*)$$

On the other hand, we have

$$V(t)=\sum_{i\in\mathcal{V}}|(x_i-\overline{x})+(\overline{x}-P_{\mathcal{X}}(\overline{x}))+(P_{\mathcal{X}}(\overline{x})-x^*)|^2\le n|P_{\mathcal{X}}(\overline{x})-x^*|^2+\varepsilon_2$$

Which together with (*) implies that

$$F(P_{\mathcal{X}}(\overline{x}))-F(x^*)\ge\frac{\delta}{2}(\frac{V(t)}{n}-\varepsilon_2) \quad (24)$$

Where $\varepsilon_2=2\overline{q}(C_1+C_2+2\varepsilon)\alpha(t)-(C_1+C_2+2\varepsilon)^2\alpha^2(t)$,

Combing (23) and (24) yields that $\forall t\ge t_{\varepsilon_2}$

$$\dot{V}(t)\le 2\sum_{i\in\mathcal{V}}\alpha(t)(f_i(x^*)-f_i(P_{\mathcal{X}}(\overline{x})+\varepsilon_1)+2\sum_{i\in\mathcal{V}}\alpha(t)L\theta(t)\alpha(t)\sqrt{mV_i(t)}+\alpha(t)4\theta(t)(\beta+\gamma)\sqrt{mnV_i(t)}$$

$$\le\alpha(t)(-\frac{\delta}{n}V(t)+2n\varepsilon_1+\delta\varepsilon_2)+2\alpha(t)L\theta(t)\alpha(t)\sqrt{mnV(t)}+\alpha(t)4(\beta+\gamma)\theta(t)\sqrt{mnV(t)}$$

Where we make use of $\sum_{i\in\mathcal{V}}\sqrt{V_i(t)}\le\sqrt{nV(t)}$. Finally, so when t becomes sufficiently large, we would discuss following inequation:

$$\dot{V}(t)\le\alpha(t)(-\frac{\delta}{n}V(t)+2L\varepsilon_3\sqrt{mnV(t)}+4(\beta+\gamma)(\hat{\theta}+\varepsilon)\sqrt{mnV(t)}+2n\varepsilon_1+\delta\varepsilon_2)$$

And next,

$$\dot{V}(t)\le\alpha(t)(-\frac{\delta}{n}V(t)+4(\beta+\gamma)(\hat{\theta}+\varepsilon_0)\sqrt{mn}\sqrt{V(t)}+\varepsilon_4)$$

Where $\varepsilon_0=\varepsilon+\dfrac{L\varepsilon_3}{2(\beta+\gamma)},\varepsilon_4=2n\varepsilon_1+\delta\varepsilon_2$

And analysis $V(t)$ by above inequality, we would show that $\liminf V(t)\le\overline{V}$, where $\overline{V}=mn^3(\dfrac{2(\beta+\gamma)(\hat{\theta}+\varepsilon_0)+\sqrt{4(\beta+\gamma)^2(\hat{\theta}+\varepsilon_0)^2+n\delta\varepsilon_4}}{\delta})^2$. we assume that $\liminf V(t)\ge\overline{V}$, which implies the existence of $\rho>0$,and $\forall t\ge t_\rho, V(t)\ge(\sqrt{\overline{V}(t)}+\rho)^2$,then from (34) one can obtain that $\forall t\ge t_\rho, \dot{V}(t)\le\alpha(t)(-\dfrac{\delta}{n}\rho^2-2\sqrt{4mn^3(\beta+\gamma)^2(\hat{\varepsilon}+\varepsilon_0)^2+n\delta\varepsilon_4\rho})$ .Since $\int_0^\infty\alpha(t)=\infty$, $V(t)$ will be negative when t is sufficiently large, which leads to a contradiction.

From the right hand side of (34), we can know the $V(t)$ never escape $[0,\overline{V}(t)]$.which implies that $\limsup V(t)\le\dfrac{16mn^3(\beta+\gamma)^2\hat{\theta}^2}{\delta^2}$. Eventually , we have $\limsup_{t\to\infty}V(t)=n\limsup_{t\to\infty}|x_i-x^*|$. ○

4. Conclusion

In this paper, we hope that the controller update is less on the distributed optimization problem through the redesign of the controller on the basis of the article [19]. We would achieve less update by completely sampling the state in the controller. In the new controller algorithm, we prove that the states converge to the constraint set, and the states is gradually consistent , finally ,the global optimal is achieved.

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