Application of Generalized Discrepancy Principles to Tikhonov Regularization for Solving Nonlinear Ill-Posed Problems

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Abstract

In this paper we consider nonlinear ill-posed problems in a Hilbert space setting. We propose a class of principle strategies for Tikhonov regularization that lead to optimal convergence rates toward the minimal-norm, least-squares solution of an ill-posed nonlinear operator equation in the presence of noisy data. Our results cover the special case of discrepancy principle of Tikhonov regularization and extend recent results of the convergence rate. In addition, we give conditions that guarantee the convergence rate $O(\delta^{\frac{2q-p+2}{2(q+1)}})$ and $O(\delta^{\frac{p}{q+1}})$ for the regularized solutions, respectively, where δ is a norm bound for the noise in the data.

Keywords

Nonlinear ill-posed problems; Generalized discrepancy principles; Convergence and rates of convergence.

1. Introduction

This paper is devoted to a study of non-linear ill-posed problems

$$F(x)=y_0 \tag{1}$$

especially of their treatment by Tikhonov regularization; here $F:D(F)\subset X\rightarrow Y$ is a non-linear operator between Hilbert spaces X and Y.

As a notion of a "solution" of problem (l.l), we choose the concept of an x^*-minimum-norm least-squares solution x_0, i.e.:

$$F(x_0)=y_0 \tag{2}$$

and

$$\| x_0-x^*\|=\min\{\| x-x^*\|: F(x)=y_0, x\in D(F)\}. \tag{3}$$

A solution of (2), (3) need not exist and, even if it does, it need not be unique, since F is non-linear. In the ensuing discussion we assume existence of an x^*-minimum-norm least-squares solution for the data $y_0\in Y$. In addition, the choice of x^* is of course crucial. An available priori information about the location of least-squares solutions (defined by (2)) has to enter into the selection of x^* In the case of multiple least squares solutions, x^* plays the role of a selection criterion. By the choice of x^* we can influence which (least-squares) solution we want to approximate.

Throughout the paper we assume that

$$\| x_0 - x^* \| = \min\{\| x - x^* \| : F(x) = y_0, x \in D(F)\}. \quad (3)$$

(i). F is continuous and Frechet differentiable with convex domain $D(F)$ and $y_0 \in R(F)$.

(ii). F is weakly (sequentially) closed, i.e. for any sequence $x_n \subset D(F)$, weak convergence of x_n to x in X and weak convergence of $F(x_n)$ to y in Y imply $x \in D(F)$ and $F(x) = y$. (Weak convergence and norm convergence will always be denoted by "$\mapsto$" and $\to$ respectively.)

Now for exact data y_0, the existence of x^*-MNS x_0 of (1) can be guaranteed. We also assume that x_α^δ, x_α are interior points of $D(F)$ for each $\alpha > 0$ and sufficiently small $\delta > 0$, this guarantees that the first order necessary optimal conditions for x_α^δ and x_α are valid.

The problem of solving (1) is, in general, ill-posed. By ill-posedness, we always mean that the solutions do not depend continuously on the data. In the case of multiple solutions this is understood in the sense of multivalued mappings. To cope with the ill-posedness, problem (1) has to be regularized. A well-known and effective technique is Tikhonov regularization. In this method a solution of problem (1) is approximated by a solution of the minimization problem

$$\min_{x \in D(F)} \{\| F(x) - y_\delta \|^2 + \alpha \| x - x^* \|^2\}, \quad (4)$$

where α is regarded as the regularized parameter, x_α^δ is the Tikhonov regularization solution, and y_δ is a δ-approximation of y_0 i.e.

$$\| y_\delta - y_0 \| \le \delta. \quad (5)$$

In the linear case aspects of stability, convergence and convergence rates have been extensively studied, e.g. in [l, 3, 6-9, 12, 16, 19, 21, 24]. In the non-linear case, the role of Tikhonov regularization to stabilize parameter estimation problems has been studied in [4, 5, 13]. Weak stability and convergence for general non-linear problems have been treated in [23].

This paper is organized as follows. In Section 2 we will present a new simple selection criterion for regularized parameter and study the convergence of the Tikhonov regularization solution. In Section 3 we will show the optimality of convergence for Tikhonov regularization solution under appropriate conditions.

2. Parameter choice and convergence

Under the assumptions on F in Section 1, the Tikhonov regularization solutions x_α^δ, x_α always exist. Without loss of the generality, we assume $x^* \in D(F)$ and

$$\| F'(x^*)^*(F(x^*) - y_0) \| > 0. \quad (6)$$

To obtain the main result, we make the following assumptions:

(H_1). There exists a constant K_0 such that for every $(x, z, v) \in D(F) \times D(F) \times X$, there exists $k(x, z, v) \in X$ such that

$$(F'(x) - F'(z))v = F'(z)k(x, z, v), \quad (7)$$

where

$$\| k(x, z, v) \| \le k_0 \| v \| \ \| x - z \|. \quad (8)$$

(H_2). There exists constants K_1, K_2 such that every $(x, z, y) \in D(F) \times D(F) \times Y$, there are $l_1(x, z, y) \in Y, l_2(x, z, F'(x)^* y) \in X$ such that

$$(F'(x)^* - F'(z)^*)y = F'(z)^* l_1(x, z, y) + l_2(x, z, F'(x)^* y), \quad (9)$$

where

$$\| l_1(x,z,y)\| \le K_1 \| y\| \ \| x-z\|, \tag{10}$$

$$\| l_2(x,z,F'(x)^* y)\| \le K_2 \| F'(x)^* y\| \ \| x-z\|. \tag{11}$$

Lemma 2.1. Let the assumption H_1 be fulfilled and let

$$K_0 \| x_0 - x^* \| < 1, \tag{12}$$

then the regularization solution x_α^δ is continuous with respect to α for $\alpha \in (\alpha_0, +\infty)$, where

$$\alpha_0 = \frac{K_0^2 \delta^2}{(1-K_0 \| x_0 - x^* \|)^2}. \tag{13}$$

Proof. Refer to [25, Lemma 2.1] by choosing $\rho > \frac{K_0 \| x_0 - x^*\|}{1-K_0 \| x_0 - x^*\|}$. For actual computations, one wants to determine an appropriate value for the regularization parameter from the computations, e.g., from the residual. The discrepancy $D(\alpha, y_\delta)$ is defined as the norm of the residual, i.e.,

$$D(\alpha, y_\delta) := \| F'(x_\alpha^\delta)^* (F(x_\alpha^\delta) - y_\delta)\|$$

In [26], the parameters choice rule was defined as follows:

$$D(\alpha, y_\delta) = \frac{\delta}{\alpha^{1/2}}$$

We'll extend the parameters choice rule above and obtain generalized results. Now in the following, the generalized parameters choice rules will be defined. Now we define a function

$$f(\alpha) = \alpha \| x_\alpha^\delta - x^* \|. \tag{14}$$

We will use the solution $\alpha := \alpha(\delta)$ of the equation

$$f(\alpha) = \delta^p \alpha^{-q} \tag{15}$$

as the regularized parameter and show that the convergence of x_α^δ, where $p>0, q>0$. Therefore we have the first order necessary optimal condition for x_α^δ, as follows:

$$F'(x_\alpha^\delta)^* (F(x_\alpha^\delta) - y_\delta) + \alpha (x_\alpha^\delta - x^*) = 0. \tag{16}$$

This implies

$$f(\alpha) = \| F'(x_\alpha^\delta)^* (F(x_\alpha^\delta) - y_\delta)\|.$$

Lemma 2.2. Assume (H_1), (H_2) and (12) hold, Then the function f defined by (14) is continuous on $(\alpha_0, +\infty)$. Moreover,

$$\lim_{\alpha\to\infty} f(\alpha) = \| F'(x^*)^* (F(x^*) - y_\delta)\|, \text{ where } \ \text{p>0, q>0.}$$

Proof. The continuity of $f(\alpha)$ on $(\alpha_0, +\infty)$ is an immediate consequence of Lemma 2.1. To verify the assertion, noting that

$$\| F(x_\alpha^\delta) - y_\delta \|^2 + \alpha \| x_\alpha^\delta - x^* \|^2 \le \| F(x^*) - y_\delta \|^2,$$

from the continuity of F, we have

$$\| F(x_\alpha^\delta) - y_\delta \| \le \| F(x^*) - y_\delta \|,$$

and

$$\lim_{\alpha\to\infty} \| x_\alpha^\delta - x^* \| = 0, \quad \lim_{\alpha\to\infty} \alpha \| x_\alpha^\delta - x^* \|^2 = 0.$$

By the assumption (H_2)

$$(F'(x_\alpha^\delta)^* - F'(x^*)^*)(F(x_\alpha^\delta) - y_\delta) = F'(x^*)^* l_1(x_\alpha^\delta, x^*, F(x_\alpha^\delta) - y_\delta) + l_2(x_\alpha^\delta, x^*, F'(x_\alpha^\delta)^*(F(x_\alpha^\delta) - y_\delta)).$$

Hence, we have

$$\|(F'(x_\alpha^\delta)^* - F'(x^*)^*)(F(x_\alpha^\delta) - y_\delta)\| \le K_1 \| F'(x^*)^*\| \ \| x_\alpha^\delta - x^*\| \ \| F(x^*) - y_\delta\| + K_2 \alpha \| x_\alpha^\delta - x^*\|^2.$$

Now from the continuity of F we obtain

$$\begin{aligned}\lim_{\alpha\to\infty} \| F'(x_\alpha^\delta)^*(F(x_\alpha^\delta) - y_\delta)\| &= \lim_{\alpha\to\infty} \| F'(x^*)^*(F(x_\alpha^\delta) - y_\delta)\| \\ &= \| F'(x^*)^*(F(x^*) - y_\delta)\|.\end{aligned}$$

Then the proof is complete.

Lemma 2.3. Let the assumption in Lemma 2.2 be fulfilled, then for sufficiently small $\delta > 0$ there exists $\alpha := \alpha(\delta)$ such that

$$f(\alpha(\delta)) = \alpha \| x_\alpha^\delta - x^*\| = \delta^p \alpha^{-q}. \tag{17}$$

Moreover,

$$\alpha(\delta) \ge \bar{\alpha}_0 := \min\{\frac{1}{2}, \frac{1}{\| x_0 - x^*\|^2}\} \delta^{\frac{p}{q+1}}.$$

Proof. Note that for sufficiently small $\delta > 0$, there holds $\bar{\alpha}_0 > \alpha_0$. Therefore, by using (6) and Lemma 2.2, we can derive that $f(\alpha)$ is continuous on $[\bar{\alpha}_0, +\infty)$ and $\lim\limits_{\alpha\to+\infty} f(\alpha) = +\infty.$

Since

$$\begin{aligned} f(\bar{\alpha}_0) = \bar{\alpha}_0 \| x_{\bar{\alpha}_0}^\delta - x^*\| &\le \sqrt{\bar{\alpha}_0 \delta^2 + \bar{\alpha}_0^2 \| x_0 - x^*\|^2} \\ &\le \sqrt{\frac{1 + \delta^{\frac{2q+2-p}{q+1}}}{2}} \delta^{\frac{p}{q+1}} \\ &< \delta^{\frac{p}{q+1}} \end{aligned}$$

for sufficiently small $\delta > 0$, we have

$$\bar{\alpha}_0^q f(\bar{\alpha}_0) < \delta^p,$$

which together with the continuity of $\alpha^q f(\alpha)$, gives the assertion, then the proof is complete.

Now we give the convergence result of the Tikhonov regularization solution x_α^δ.

Theorem 2.4. Let the assumptions in Lemma 2.3 be fulfilled, for each sequence δ_n such that $\delta_n \to 0$ as $n \to \infty$, let $\alpha_n := \alpha(\delta_n)$ be determined by (15) with δ replaced by δ_n and let $x_{\alpha_n}^{\delta_n}$ be the solution of (4) with y_δ replaced by y_{δ_n}, where y_{δ_n} denotes the perturbed data of y_0, $\| y_{\delta_n} - y_0\| \le \delta_n$. Then the sequence $\{x_{\alpha_n}^{\delta_n}\}$ has a convergent subsequence. The limit of every convergent subsequence is an x^*-MNS. If in addition, the $x^* -$MNS x_0 of (1) is unique, then $\lim\limits_{n\to\infty} x_{\alpha_n}^{\delta_n} = x_0$.

Proof. Suppose there is a subsequence α_{n_k} of α_n such that $\lim\limits_{k\to\infty} \alpha_{n_k} \ge \alpha_0$ for some positive constant α_0.

By virtue of

$$\alpha_{n_k} \| x_{\alpha_{n_k}}^{\delta_{n_k}} - x^*\| = \delta_{n_k}^p \alpha_{n_k}^{-q}, \text{where } p > 0, q > 0,$$

hence $\lim_{k\to\infty}\| x_{\alpha_{n_k}}^{\delta_{n_k}} - x^*\| = 0, \quad \lim_{k\to\infty}\alpha_{n_k}^q \| x_{\alpha_{n_k}}^{\delta_{n_k}} - x^*\|^2 = 0.$

It follows from assumption (H_2) that

$$\|(F'(x_{\alpha_{n_k}}^{\delta_{n_k}})^* - F'(x^*)^*)(F(x_{\alpha_{n_k}}^{\delta_{n_k}}) - y_{\delta_{n_k}})\| \le K_1 \| F'(x^*)^*\| \; \| F(x^*) - y_{\delta_{n_k}} \| \; \| x_{\alpha_{n_k}}^{\delta_{n_k}} - x^*\| + K_2 \alpha_{n_k} \| x_{\alpha_{n_k}}^{\delta_{n_k}} - x^*\|^2.$$

Therefore

$$\begin{aligned} \| F'(x^*)^*(F(x^*) - y_0)\| &= \lim_{k\to\infty} \| F'(x^*)^*(F(x_{\alpha_{n_k}}^{\delta_{n_k}}) - y_{\delta_{n_k}})\| \\ &= \lim_{k\to\infty} \| F'(x_{\alpha_{n_k}}^{\delta_{n_k}})^*(F(x_{\alpha_{n_k}}^{\delta_{n_k}}) - y_{\delta_{n_k}})\| \\ &= \lim_{k\to\infty} \delta_{n_k}^p \alpha_{n_k}^{-q} = 0, \end{aligned}$$

which is a contradiction to (6). Hence

$$\lim_{n\to\infty} \alpha_n = 0 \tag{18}$$

Notice

$$\| F(x_{\alpha_n}^{\delta_n}) - y_{\delta_n} \|^2 + \alpha_n \| x_{\alpha_n}^{\delta_n} - x^*\|^2 \le \delta_n^2 + \alpha_n \| x_0 - x^*\|^2,$$

we have

$$\| F(x_{\alpha_n}^{\delta_n}) - y_{\delta_n} \|^2 \le \delta_n^2 + \alpha_n \| x_0 - x^*\|^2, \tag{19}$$

$$\| x_{\alpha_n}^{\delta_n} - x^*\|^2 \le \| x_0 - x^*\|^2 + \frac{\delta_n^2}{\alpha_n}, \tag{20}$$

which together with Lemma 2.3 gives that $x_{\alpha_n}^{\delta_n}$ is bounded, hence $x_{\alpha_n}^{\delta_n}$ has a weakly convergent subsequence. Without loss of generality, we assume that $x_{\alpha_n}^{\delta_n} \mapsto \bar{x}$ as $n \to \infty$. From (18) and (19), we obtain $\lim_{n\to\infty} F(x_{\alpha_n}^{\delta_n}) = y_0$. By virtue of the weak closedness of F, we derive that $\bar{x} \in D(F)$ and $F(\bar{x}) = y_0$. From the weak lower semicontinuity of the Hilbert space norm and (20), we obtain

$$\| \bar{x} - x^*\| \le \liminf_{n\to\infty} \| x_{\alpha_n}^{\delta_n} - x^*\| \le \limsup_{n\to\infty} \| x_{\alpha_n}^{\delta_n} - x^*\| ,$$

$$\| \bar{x} - x^*\| \le \| x_0 - x^*\| .$$

Notice x_0 is an $x^* -$MNS, we have $\| \bar{x} - x^*\| = \| x_0 - x^*\|$. Therefore, $\bar{x}$ is an $x^* -$MNS, and

$$\lim_{n\to\infty} \| x_{\alpha_n}^{\delta_n} - x^*\| = \| x_0 - x^*\|.$$

Since

$$\| x_{\alpha_n}^{\delta_n} - \bar{x}\|^2 = \| x_{\alpha_n}^{\delta_n} - x^*\|^2 - 2(x_{\alpha_n}^{\delta_n} - x^*, \bar{x} - x^*) + \| \bar{x} - x^*\|^2,$$

we have $\quad \lim_{n\to\infty} \| x_{\alpha_n}^{\delta_n} - \bar{x}\| = 0.$

3. Rates of convergence

The purpose of this section is to show that the optimal convergence rate can be derived under suitable smoothness conditions according to the regularized parameter chosen in Section 2.

Lemma 3.1. Let the assumptions in Theorem 2.4 be fulfilled. Then, there are constants $C_1, C_2 > 0$ such that, for sufficiently small $\delta > 0$, the relation

$$C_1 \le \delta^p \alpha(\delta)^{-q-1} \le C_2 \tag{21}$$

holds.

Proof. From (14) and (16), we have

$$f(\alpha, y_\delta) = \alpha \| x_\alpha^\delta - x^* \| \text{for all } \alpha > 0, y_\delta \in Y.$$

This with Theorem 2.4, implies that

$$\lim_{\delta \to 0} (\delta^p \alpha(\delta)^{-q-1}) = \lim_{\delta \to 0} (\alpha(\delta)^{-1} f(\alpha(\delta), y_\delta)) = \| x_0 - x^* \| > 0,$$

because of (3). Then it implies the assertion.

Lemma 3.2. Let the assumption in Lemma 2.3 be fulfilled and let $\alpha := \alpha(\delta)$ be determined by (15). If $x_0 - x^* \in R(F'(x_0)^* F'(x_0))$ and $(K_0 + K_2) \| x_0 - x^* \| < 1$, then there exist constants $C_3 > 0$ and $\delta_0 > 0$ such that for all $0 < \delta \le \delta_0$, there holds $\| x_{\alpha(\delta)} - x_0 \| \le C_3 \alpha(\delta)$

Proof. Refer to [26, Lemma 4] by choosing $f(\alpha(\delta)) = \delta^p \alpha^{-q}$.

Lemma 3.3. Let the assumption (H_1) hold and let $K_0 \| x_0 - x_* \| < 1$, then for each $\alpha > 0$ and $\delta > 0$, there holds $\| x_{\alpha(\delta)} - x_\alpha \| \le C_4 \dfrac{\delta}{\sqrt{\alpha}}$, where

$$C_4 = \sqrt{\frac{4}{1 - K_0 \| x_0 - x_* \|}}.$$

Proof. Refer to [26, Lemma 5].

Theorem 3.4. Let the assumptions $(H_1), (H_2)$ hold, let $\alpha(\delta)$ be determined by (15). If $x_0 - x^* \in R(F'(x_0)^* F'(x_0))$ and $(K_0 + K_2) \| x_0 - x^* \| < 1$.

Then the following statements are true.

(i). If $p - 1 = q > 0$, then $\| x_{\alpha(\delta)}^\delta - x_0 \| = O(\delta^{\frac{1}{2}})$, as $\delta \to 0$.

(ii). If $\dfrac{3}{2} p - 1 = q > 0$, then $\| x_{\alpha(\delta)}^\delta - x_0 \| = O(\delta^{\frac{2}{3}})$, as $\delta \to 0$.

Proof. (i). Lemma 3.1 and 3.2 are applicable and yields, together with by now standard estimates of Lemma 3.3, for suitably small $\delta > 0$,

$$\begin{aligned} \| x_{\alpha(\delta)}^\delta - x_0 \| &\le \| x_{\alpha(\delta)}^\delta - x_{\alpha(\delta)} \| + \| x_{\alpha(\delta)} - x_0 \| \\ &\le C_3 \alpha(\delta) + C_4 \delta \alpha^{-\frac{1}{2}} \\ &\le C_3 C_1^{-\frac{1}{p}} \delta + C_4 C_2^{\frac{1}{2p}} \delta^{\frac{1}{2}} \\ &= O(\delta^{\frac{1}{2}}), \end{aligned}$$

with suitable constants $C_3, C_4 > 0$ and C_1, C_2 as in Lemma 3.1.

(ii). Similarly as above, Lemma 3.1, 3.2 and 3.3 yield

$$\begin{aligned} \| x_{\alpha(\delta)}^\delta - x_0 \| &\le \| x_{\alpha(\delta)}^\delta - x_{\alpha(\delta)} \| + \| x_{\alpha(\delta)} - x_0 \| \\ &\le C_3 \alpha(\delta) + C_4 \delta \alpha^{-\frac{1}{2}} \\ &\le C_3 C_1^{-\frac{2}{3p}} \delta^{\frac{2}{3}} + C_4 C_2^{\frac{1}{3p}} \delta^{\frac{2}{3}} \\ &= O(\delta^{\frac{2}{3}}), \end{aligned}$$

with suitable constants $C_3, C_4 > 0$ and C_1, C_2 as in Lemma 3.1.

Similarly, we can prove

Theorem 3.5. Let the assumptions $(H_1),(H_2)$ hold, let $\alpha(\delta)$ be determined by (15). If $x_0 - x^* \in R(F'(x_0)^* F'(x_0))$ and $(K_0 + K_2)\| x_0 - x^*\| < 1$.

Then the following statements are true.

(i). If $\frac{p}{q+1} > \frac{2}{3}$, then $\| x^{\delta}_{\alpha(\delta)} - x_0\| = O(\delta^{\frac{2q-p+2}{2(q+1)}})$, as $\delta \to 0$.

(ii). If $0 < \frac{p}{q+1} \le \frac{2}{3}$, then $\| x^{\delta}_{\alpha(\delta)} - x_0\| = O(\delta^{\frac{p}{q+1}})$, as $\delta \to 0$.

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