A Modified Iteration Method for Solving Nonlinear Ill-Posed Problems

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Abstract

In this paper we consider a new iteration method for solving nonlinear ill-posed problems and propose a stopping rule for perturbed data with noise level δ. Under certain conditions, we obtain stable iteration, convergence, and rates of convergence.

Keywords

Nonlinear ill-posed problems; Discrepancy principles; Convergence and rates of convergence.

1. Statement of the problem

With the growing interest in the applied sciences, a lot of attention has been paid to the study of nonlinear ill-posed problems, that is, problems which can be formulated as a nonlinear operator equation:

$$F(x) = y, \tag{1}$$

where $F: D(F) \to Y$ with domain $D(F) \subset X$. We restrict our attention to Hilbert spaces X and Y with inner products $(\cdot,\cdot)$ and norms $\|\cdot\|$ respectively. In general, problem (1) is ill-posed in the sense that the solution of (1) does not depend continuously on the right hand side, which is often obtained by measurement and hence contains error. Let us assume throughout that (1) has a solution x_*, and that we have approximate y^δ with

$$\| y^\delta - y\| \le \delta. \tag{2}$$

We are mainly interested in problems of the form (1) for which the solution x_* does not depend continuously on the right hand side data y. Such ill-posed problems need to be regularized to obtain reasonable approximations to x_* Tikhonov regularization is the most well known method for solving nonlinear ill-posed problems, and it has received a lot of attention in recent years (cf.[2, 5, 6, 14, 15]). Iteration methods are also attractive since they are straightforward to implement for the numerical solution of nonlinear ill-posed problems. In [4], Landweber iteration [7] was extended to the study of nonlinear problems. In this paper, we contribute to the study of a finite dimensional approximation of iteration for nonlinear operator equations (1) ; the novel method we consider below requires a locally uniformly bounded Frechet derivative $F'(\cdot)$ of F, and is defined via the adjoint $F'(\cdot)^*$ of $F'(\cdot)$ as follows:

$$x_{k+1} = x_k + F'(x_k)^* F'(x_k)(y - F(x_k)), k = 0,1,2,\cdots, \tag{3}$$

where x_0 is is an initial guess which may incorporate a priori knowledge of an exact solution x_*. If the iteration is applied to the perturbed problem with y^δ instead of y in (3), then we rewrite x_k^δ

instead of x_k for the iterates; we will always assume that $x_0^\delta = x_0$. We emphasize that for a fixed number of iterations the process (1.3) is a stable algorithm, even if y^δ does not belong to the range of F.

We believe that in many practical examples it is almost impossible to check analytically. In addition, It is easy to check the following local property in a ball $R_\rho(x_0)$ of radius ρ around x_0:

$$\| F(x)-F(\tilde{x})-F'(x)^* F'(x)(x-\tilde{x})\| \le \eta \| F(x)-F(\tilde{x})\|, \tag{4}$$

where $\eta < \frac{1}{2}, x, \tilde{x} \in R_\rho(x_0) \subset D(F)$. If y^δ does not belong to the range of F, then the iterates x_k^δ of (1.3) cannot converge but still allow a stable approximation of x_* provided the iteration is stopped after $k_* = k_*(\delta)$ steps according to a generalized discrepancy principle, i.e.,

$$\| y^\delta - F(x_{k_*}^\delta)\| \le \tau\delta < \| y^\delta - F(x_k^\delta)\|, 0 \le k \le k_*, \tag{5}$$

where τ is a positive number depending on η of (1.4), i.e.,

$$\tau > \frac{2(1+\eta)}{1-2\eta} > 2 \tag{6}$$

In other words, k_* is one of the first indices for which the size of the residual $y^\delta - F(x_k^\delta)$ has about the order of the data error. In this paper, we contribute to the study of stable method for solving nonlinear ill-posed problems. The rest of the paper is organized as follows: In Section 2 we point out that iteration (3) is well defined. Section 3 contributes to rates of convergence which can be derived if the sought solution admits some smoothness conditions.

2. Convergence of the iteration

As in the linear case the Landweber iteration can only converge if problem (1) is properly scaled. For our analysis we assume that

Assumption 2.1. (i) Problem (1) is properly scaled, i.e., F is Frechet derivative in $R_\rho(x_0)$, and the Frechet derivative $F'(x)$ at $x \in R_\rho(x_0)$ satisfies

$$\| F'(x)\| \le 1, x \in R_\rho(x_0) \subset D(F) \tag{7}$$

(ii) There is a constant C such that forever pair $x, \tilde{x} \in R_\rho(x_0)$ there holds

$$\| F(x)-F(\tilde{x})-F'(x)^* F'(x)(x-\tilde{x})\| \le C \| x-\tilde{x}\| \ \| F(x)-F(\tilde{x})\|, \tag{8}$$

where $\| x-\tilde{x}\|$ is sufficiently small.

Lemma 2.1. If (4) holds and if x_* is a solution of (1) in $R_\rho(x_0)$, then any other solution $\tilde{x}_*$ in $R_\rho(x_0)$ fulfills $x_* - \tilde{x}_* \in N(F'(x_*)^* F'(x_*))$, and vice versa. $N(\cdot)$ denotes the null space of an operator.

Proof. It follows immediately from (4) that

$$\frac{1}{1+\eta}\| F'(x)^* F'(x)(x-\tilde{x})\| \le \| F(x)-F(\tilde{x})\| \le \frac{1}{1-\eta}\| F'(x)^* F'(x)(x-\tilde{x})\|, \tag{9}$$

holds for all $x, \tilde{x} \in R_\rho(x_0)$ This implies the assertion.

Lemma 2.2. Assume that x_* is a solution of (1) in $R_{\rho/2}(x_0)$, and denote by k_* the termination index of the iteration according to the stopping rule (5), (6) for the case of perturbed data y^δ satisfying (2). If (4) and (8) hold, then we have

$$\| x_* - x_{k+1}^\delta \| \le \| x_* - x_k^\delta \|, 0 \le k \le k_*; \tag{10}$$

and if $\delta = 0$,

$$\sum_{k=1}^{\infty} \| y - F(x_k) \|^2 < \infty. \tag{11}$$

Proof. Let $0 \le k < k_*$. Exploiting (3), (4) and (7), we obtain by induction that $x_k^\delta \in R_{\rho/2}(x_*) \subset R_\rho(x_0)$ and that

$$\begin{aligned} \| x_* - x_{k+1}^\delta \|^2 - \| x_* - x_k^\delta \|^2 &= 2(x_k^\delta - x_*, x_{k+1}^\delta - x_k^\delta) + \| x_{k+1}^\delta - x_k^\delta \|^2 \\ &= 2(y^\delta - F(x_k^\delta), F(x_*) - F(x_k^\delta) - F'(x_k^\delta)^* F'(x_k^\delta)(x_* - x_k^\delta) + y^\delta - y) \\ &\quad - (y^\delta - F(x_k^\delta), (I - (F'(x_k^\delta)^* F'(x_k^\delta))^2)(y^\delta - F(x_k^\delta))) - \| y^\delta - F(x_k^\delta) \|^2 \\ &\le \| y^\delta - F(x_k^\delta) \| ((2\eta - 1) \| y^\delta - F(x_k^\delta) \| + 2(1+\eta)\delta) \end{aligned}$$

Since $k < k_*$ the right hand side is negative because of (5), and we have verified (10). If $\delta = 0$, then we have actually verified the sharper inequality $\| x_* - x_{k+1} \|^2 + (1-2\eta) \| y - F(x_k) \|^2 \le \| x_* - x_k \|^2$, valid for all $k \in N_0$. By introduction, we obtain $\sum_{n=1}^{\infty} \| y - F(x_k) \|^2 \le \frac{1}{1-2\eta} \| x_* - x_0 \|^2$, and assertion (11) follows. We remark that if $\delta \ne 0$ then we can show in a similar way that

$$\sum_{k=0}^{k_*-1} \| y^\delta - F(x_k^\delta) \|^2 \le \frac{\tau}{(1-2\eta)\tau - 2(1+\eta)} \| x_* - x_0 \|^2 \tag{12}$$

Theorem 2.3. If (4) and (12) are satisfied and if (1) is solvable in $R_{\rho/2}(x_0)$, then x_k converges to a solution $x_* \in R_{\rho/2}(x_0)$ of (1). If x^+ denotes the unique solution of minimal distance to x_0, and if in addition $N(F'(x^+)^* F'(x^+)) \subset N(F'(x)^* F'(x))$ for all $x \in R_\rho(x_0)$, then x_k converges to x^+.

Proof. Let $\tilde{x}_*$ be any solution of (1) in $R_{\rho/2}(x_0)$, and put $e_k := \tilde{x}_* - x_k$. From Lemma 2.2, it follows that] $\| e_k \|$ is monotonically decreasing to some $\varepsilon \ge 0$.

We show next that e_k is a Cauchy sequence. For $j \ge k$ we choose l with $j \ge l \ge k$ such that

$$\| y - F(x_l) \| \le \| y - F(x_i) \|, k \le i \le j. \tag{13}$$

We have

$$\| e_j - e_k \| \le \| e_j - e_l \| + \| e_l - e_k \|. \tag{14}$$

and

$$\begin{aligned} \| e_j - e_l \|^2 &= 2(e_l - e_j, e_l) + \| e_j \|^2 - \| e_l \|^2, \\ \| e_l - e_k \|^2 &= 2(e_l - e_k, e_l) + \| e_k \|^2 - \| e_l \|^2. \end{aligned} \tag{15}$$

For $k \to \infty$, the last two terms on each of the right-hand sides of (15) converge to $\varepsilon^2 - \varepsilon^2 = 0$. We now apply (3) and (9) to show that $(e_l - e_k, e_l)$ also tends to zero as $k \to \infty$:

$$
\begin{aligned}
|(e_l - e_k, e_l)| &= |\sum_{i=k}^{l-1} (F'(x_i)^* F'(x_i)(y - F(x_i)), e_l| \\
&\le |\sum_{i=k}^{l-1} (y - F(x_i), F'(x_i)^* F'(x_i) e_l)| \\
&\le \sum_{i=k}^{l-1} \|y - F(x_i)\| \ \| F'(x_i)^* F'(x_i)(\tilde{x}_* - x_l)\| \\
&\le (1+\eta)\sum_{i=k}^{l-1} \|y - F(x_i)\| \ \| y - F(x_l)\| + 2\eta \sum_{i=k}^{l-1} \|y - F(x_i)\|^2 \\
&\le (1+3\eta)\sum_{i=k}^{l-1} \|y - F(x_i)\|^2,
\end{aligned}
$$

where we have used (12) to obtain the last inequality. Similarly, one can show that

$$|(e_j - e_l, e_l)| \le (1+3\eta)\sum_{i=l}^{j-1} \|y - F(x_i)\|^2.$$

With these estimates, it follows from (11) that the right-hand sides of (15) go to zero as $k \to \infty$, and then e_k and x_k are Cauchy sequences. We denote by x_* the limit of x_k and observe that x_* is a solution of (1) because the residuals $y - F(x_k)$ converge to zero as $k \to \infty$.

It follows from Lemma 2.1, (1) has a unique solution x^+ of minimal distance to x_0, which satisfies $x^+ - x_0 \in N(F'(x^+)^* F'(x^+))^{\perp}$.

If $N(F'(x^+)^* F'(x^+)) \subset N(F'(x_k)^* F'(x_k))$, for all $k = 0, 1, 2, \cdots$, then it is clear that

$$x_k - x_0 \in N(F'(x^+)^* F'(x^+))^{\perp}, k = 1, 2, \cdots,$$

hence, $x^+ - x_* = x^+ - x_0 + x_0 - x_* \in N(F'(x^+)^* F'(x^+))^{\perp}$.

This together with Lemma 2.1 implies that $x_* = x^+$.

Theorem 2.4. Under the assumptions of Theorem 2.3, if y^δ fulfills (2), and if the perturbed iteration is stopped with $k_*(\delta)$ according to the discrepancy principle (5), (6), then $x^\delta_{k_*(\delta)} \to x_*, \delta \to 0$.

Proof. Let $\delta_n, n = 1, 2, \cdots$, be a sequence converging to zero as $n \to \infty$, and let $y_n := y^{\delta_n}$ be a corresponding sequence of perturbed data. For each pair (δ_n, y_n) denote by $k_n = k_*(\delta_n)$ the corresponding stopping index determined from the discrepancy principle (5), (6).

Assume first that k is a finite accumulation point of k_n. Without loss of generality we can assume that $k_n = k$ for all $n \in N$. Thus, from the definition of k_n it follows that

$$\| y_n - F(x_k^{\delta_n})\| \le \tau \delta_n. \tag{16}$$

Since x_k^δ depends continuously on y^δ as k is fixed now, we also have

$$x_k^{\delta_n} \to x_k, F(x_k^{\delta_n}) \to F(x_k), n \to \infty. \tag{17}$$

Taking the limit in (16) yields $F(x_k) = y$. Thus, $x_k = x_*$ by Theorem 2.3, and with (17) we obtain $x_k^{\delta_n} \to x_*, n \to \infty$. It remains to consider the case where $k_n \to \infty, n \to \infty$. Without loss of generality we assume that k_n increases monotonically with n. Then, for $n > m$ we conclude from Lemma 2.2:

$$
\begin{aligned}
\| x_{k_n}^{\delta_n} - x_*\| &\le \| x_{k_{n-1}}^{\delta_n} - x_*\| \le \cdots \| x_{k_m}^{\delta_n} - x_*\| \\
&\le \| x_{k_m}^{\delta_n} - x_{k_m} \| + \| x_{k_m} - x_*\|.
\end{aligned}
\tag{18}
$$

From Theorem 2.3 we deduce that we can fix m so large that the last term on the right-hand side of (18) is sufficiently close to zero; now that k_m is fixed, we can apply (17) to conclude that the left-hand side of (18) must go to zero as $n \to \infty$, and the proof is complete.

3. Rates of convergence

In Section 2 we have considered the convergence of $x_{k_n}^{\delta_n}$, but we cannot obtain any information on the rates of convergence. In fact, the rates of convergence can be arbitrarily slow. Therefore, to guarantee a suitable rate, some additional assumptions should be imposed on x^+ and these conditions are called ”source conditions.” The following one is frequently used in nonlinear ill-posed problems: there is a $\upsilon > 0$ and an element $f \in X$ such that

$$x^+ - x_0 = (F'(x^+)^* F'(x^+))^{\upsilon} f \tag{19}$$

In contrast to Tikhonov regularization, assumption (19) is not enough to obtain convergence rates for the proposed iteration; to proceed, an additional properties of F: we require

$$F'(x) = R_x (F'(x^+)^* F'(x^+)), \tag{20}$$

where $\{R_x : x \in R_\rho(x_0)\}$ is a family of bounded linear operators $R_x : Y \to Y$ with

$$\| R_x - I \| \le C \| x - x^+ \|, x \in R_\rho(x_0), \tag{21}$$

and C is a positive constant. Note that in the linear case $R_x \equiv I$; therefore,(20) may be interpreted as a further restriction of the “nonlinearity” of F. In particular, (20) implies that $N(F'(x^+)^* F'(x^+)) \subset N(F'(x)^* F'(x)), x \in R_\rho(x_0)$. It is not difficult to see that (20), (21) imply (8) and thus (4) with $\tilde{x} = x^+$ for ρ sufficiently small, since for $x \in R_\rho(x_0)$

$$\begin{aligned} \| F(x) - F(x^+) - F'(x)^* F'(x)(x - x^+) \| &= \| \int_0^1 (F'(z_t) - F'(x)^* F'(x))(x - x^+) dt \| \\ &\le \| (R_{z_t} - I + I - R_x) F'(x^+)^* F'(x^+)(x - x^+) \| \, dt \\ &\le \frac{3}{2} C \| F'(x^+)^* F'(x^+)(x - x^+) \| \ \| (x - x^+) \| \end{aligned} \tag{22}$$

holds, where $z_t := tx + (1-t)x^+, 0 \le t \le 1$.

Theorem 3.1. Assume that problem (1) has a solution in $R_{\rho/2}(x_0)$, that y^δ satisfies (2) and that F fulfills (4), (7), (20) and (21). If $x^+ - x_0$ satisfies (19) with some $0 < \upsilon \le 1$ and $\| f \|$ sufficiently small, then there exists a positive constant c_*, depending on υ only, with

$$\| x^+ - x_k^\delta \| \le c_* \| f \| (k+1)^{-\upsilon} \tag{23}$$

$$\| y^\delta - F(x_k^\delta) \| \le c_* \| f \| (k+1)^{-\upsilon - 1} \tag{24}$$

for $0 \le k < k_*$. Here, as before, k_* is the stopping index of the discrepancy principle (5), (6). In the case of exact data, (23) and (24) hold for all $k \ge 0$.

Proof. By Lemma 2.2, the iteration (3) is well defined, since all iterates $x_k^\delta, 0 \le k \le k_*$, remain in $R_\rho(x_0) \subset D(F)$. Moreover, by (12), the stopping index k_* is finite for $\delta > 0$. To simplify the notation we put $K := F'(x^+)$ and $e_k := x^+ - x_k^\delta$, the error of the k th iterate x_k^δ. Given $0 \le k < k_*$, we obtain from (3) the representation

$$\begin{aligned} e_{k+1} &= e_k - K^*(F(x^+) - F(x_k^\delta)) + (K^* - F'(x_k^\delta)^* F'(x_k^\delta))(y^\delta - F(x_k^\delta)) + K^*(y - y^\delta) \\ &= (I - K^*K)e_k - K^*(F(x^+) - F(x_k^\delta) - K(x^+ - x_k^\delta)) + K^*(I - R_{x_k^\delta}^*)(y^\delta - F(x_k^\delta)) \\ &\quad + K^*(y - y^\delta). \end{aligned} \tag{25}$$

Similar to (22),

$$\| F(x^+) - F(x_k^\delta) - K^*K(x^+ - x_k^\delta)\| \le \frac{1}{2} C \| e_k \| \ \| K^*Ke_k \|.$$

On the other hand, for $0 \le k < k_*$, note that $\tau > 2$, and the triangle inequality imply that

$$\| y^\delta - F(x_k^\delta)\| \le 2(\| y^\delta - F(x_k^\delta)\| - \delta) \le 2\| y - F(x_k^\delta)\|, \tag{26}$$

hence, by (21) and (9), note that $\eta < \frac{1}{2}$,

$$\|(I - R_{x_k^\delta}^*)(y^\delta - F(x_k^\delta))\| \le 4C \| e_k \| \ \| K^*Ke_k \|.$$

Consequently, e_k satisfies the following inhomogeneous difference equation

$$e_{k+1} = (I - K^*K)e_k + K^*z_k + K^*(y - y^\delta),$$

with

$$\| z_k \| \le \frac{9}{2} C \| e_k \| \ \| K^*Ke_k \|, 0 \le k < k_*. \tag{27}$$

For $0 \le k \le k_*$, this yields the "closed expression" for the error

$$e_k = (I - K^*K)^k e_0 + \sum_{j=0}^{k-1} (I - K^*K)^j K^* z_{k-j-1} + \sum_{j=0}^{k-1} (I - K^*K)^j K^*(y - y^\delta), \tag{28}$$

and consequently

$$K^*Ke_k = (I - K^*K)^k K^*Ke_0 + \sum_{j=0}^{k-1} (I - K^*K)^j (K^*K)^{\frac{3}{2}} z_{k-j-1} + \sum_{j=0}^{k-1} (I - K^*K)^j K^*KK^*(y - y^\delta).$$

We now want to show that

$$\begin{aligned} \| e_j \| &\le c_* \| f \| (j+1)^{-\upsilon}, \\ \| K^*Ke_j \| &\le c_* \| f \| (j+1)^{-\upsilon-1} \end{aligned} \tag{29}$$

hold for all $0 \le j < k_*$ with $c_* = 2[1 + \frac{2(2\eta - 1)}{8\eta^2 - 8\eta - 1}]$.

For $j = 0$ (29) is always true; for $j > 0$ the proof is done by induction: we assume that (29) is true for all $0 \le j < k$ with some $k < k_*$, and we have to verify (29) for $j = k$. Since $\| K \| \le 1$ by assumption, we have, cf., e.g., [16, 17],

$$\|\sum_{j=0}^{k-1} (I - K^*K)^j K^* \| \le \sqrt{k}, \quad \|(I - K^*K)^k (K^*K)^\upsilon \| \le (k+1)^{-\upsilon},$$

$$\|(I - K^*K)^j K^* \| \le (j+1)^{-\frac{1}{2}}, \quad \|(I - K^*K)^j KK^* \| \le (j+1)^{-1}.$$

With these bounds and (28) we obtain

$$\| e_k \| \le (k+1)^{-\upsilon} \| f \| + \sum_{j=0}^{k-1} (j+1)^{-\frac{1}{2}} \| z_{k-j-1} \| + \sqrt{k}\delta.$$

We apply (27) and (29) to estimate the sum on the right-hand side:

$$\sum_{j=0}^{k-1}(j+1)^{-\frac{1}{2}}\| z_{k-j-1}\| \quad\le\quad \frac{9}{2}Cc_*^2\| f\|^2(k+1)^{-2\upsilon}\sum_{j=0}^{k-1}(\frac{j+1}{k+1})^{-\frac{1}{2}}(\frac{k-j}{k+1})^{-2\upsilon-\frac{1}{2}}\frac{1}{k+1}.$$

The sum on the right-hand side, denoted by S_k, can be estimated from above by

$$S_k\le\int_h^{1-h}s^{-\frac{1}{2}}(1-s)^{-2\upsilon-\frac{1}{2}}ds=\begin{cases} O(1), & 0<\upsilon<\frac{1}{4},\\ O(ln(k+1)), & \upsilon=\frac{1}{4},\\ O((k+1)^{2\upsilon-\frac{1}{2}}), & \frac{1}{4}<\upsilon\le\frac{1}{2},\end{cases}$$

and hence,

$$\sum_{j=0}^{k-1}(j+1)^{-\frac{1}{2}}\| z_{k-j-1}\|\le C_\upsilon(k+1)^{-\upsilon}\| f\|^2,$$

where $C_\upsilon>0$ depends on $\upsilon\in(0,\frac{1}{2}]$ but is independent of $\| f\|, k$ and δ. Finally, we obtain $\| e_k\|\le(1+C_\upsilon\| f\|)(k+1)^{-\upsilon}+\sqrt{k}\delta.$

Similarly, one can prove that

$$\| K^*Ke_k\|\le(1+C_\upsilon\| f\|)(k+1)^{-\upsilon-1}\| f\|+(k+1)^{-\frac{1}{2}}\delta \tag{30}$$

Because of (9) and (5), (6) we have

$$2\frac{1+\eta}{1-2\eta}\delta<\| y^\delta-F(x_k^\delta)\|\le\delta+\frac{1}{1-\eta}\| K^*Ke_k\|$$

Together with (30) this yields

$$\frac{8\eta^2-8\eta-1}{2(2\eta-1)}\delta\le(1+C_\upsilon\| f\|)(k+1)^{-\upsilon-1}\| f\| \tag{31}$$

Combining these estimates, we arrive at

$$\| e_k\|\le c\| f\|)(k+1)^{-\upsilon},\quad \| K^*Ke_k\|\le c\| f\|(k+1)^{-\upsilon-1},$$

with $c=[1+\frac{2(2\eta-1)}{8\eta^2-8\eta-1}](1+C_\upsilon\| f\|)).$

Now, if $\| f\|$ is sufficiently small, namely if $C_\upsilon\| f\|\le 1,$ then $c\le c_*,$ and (29) follows for $j=k;$ thus, (23) has been verified. Assertion (24) follows from (29) by means of (26) and (9).

Theorem 3.2. Under the assumptions of Theorem 3.1 we have

$$k_*\le c_1(\frac{\| f\|}{\delta})^{\frac{1}{\upsilon+1}}, \| x^+-x_{k_*}^\delta\|\le c_2\| f\|^{\frac{1}{2\upsilon+1}}\delta^{\frac{2\upsilon}{2\upsilon+1}}, \tag{32}$$

with some constants $c_1,c_2>0,$ depending on υ only.

Proof. We use the same notation as in the proof of Theorem 3.1. By (28) we can write

$$e_{k_*}=(K^*K)^\upsilon f_{k_*}+\sum_{j=0}^{k_*-1}(I-K^*K)^jK^*(y-y^\delta), \tag{33}$$

where

$$f_{k_*} = (I - K^*K)^{k_*} f + \sum_{j=0}^{k_*-1} (I - K^*K)^j (K^*K)^{\frac{1}{2}-\upsilon} \tilde{z}_{k_*-j-1},$$

and $\| \tilde{z}_j \| = \| z_j \|,\ j = 0,1,2,\cdots,k_* - 1,$

As in the proof of Theorem 3.1 we obtain

$$\| f_{k_*} \| \le \| f \| + \frac{9}{2} C c_*^2 \| f \|^2 \sum_{j=0}^{k_*-1} (j+1)^{\upsilon-\frac{1}{2}} (k_* - j)^{-2\upsilon-\frac{1}{2}},$$

where $\tilde{C}_\upsilon$ is a finite, positive number which depends on υ but not on k_*. As $\| f \|$ has to be small anyway, there is no loss of generality in restricting $\| f \| \le 1$. Hence,

$$\| f_{k_*} \| \le (1 + \tilde{C}_\upsilon) \| f \|. \tag{34}$$

On the other hand, from (9) we have

$$\begin{aligned} \| K(K^*K)^\upsilon f_{k_*} \| &= \| K e_{k_*} - [I - (I - KK^*)^{k_*}](y - y^\delta) \\ &\le (1+\eta) \| y - F(x_{k_*}^\delta) \| + \delta \\ &\le ((1+\eta)(1+\tau)+1)\delta. \end{aligned}$$

Thus, together with (34), the interpolation inequality yields

$$\| (K^*K)^\upsilon f_{k_*} \| \le C \| f \|^{\frac{1}{2\upsilon+1}} \delta^{\frac{2\upsilon}{2\upsilon+1}}$$

for some constant $C > 0$.

From (33) we conclude

$$\| e_{k_*} \| \le \| (K^*K)^\upsilon f_{k_*} \| + \sqrt{k_*}\,\delta, \tag{35}$$

and thus the assertion is proved if $k_* = 0$. Otherwise, we apply (31) with $k = k_* - 1$ to obtain $k_*^{\nu+1} \le c_* \frac{\| f \|}{\delta}$, with c_* as in Theorem 3.1; this yields (32), and the error estimate now follows again from (35), and the proof is complete.

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