Hybrid Finite Difference Scheme for Steady Heat Conduction Equation

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Abstract

The steady heat equation is an importrant equantion in heat transfer,which is widely used in a large number of enginering practice. At the same time,it is a class of elliptic partial equation, however, like many other partial differential equations(PDE),exact solutions are difficult to obtain even for simple geometries. The finite difference method (FDM) is one simple algorithm. In this paper, a hybrid finite difference scheme for the numerical solution of 2D heat conduction equation is proposed. The accuracy and efficiency of the schemes are confirmed by two two examples.

Keywords

Heat conduction equation, finite difference method, finite difference scheme.

1. Introduction

Differential equations are equations that involve an unknown function and derivatives, Partial differential equations(PDE) include elliptic differential equations, parabolic differential equations, hyperbolic differential equations.These equations describe the behavior of many engineering phenomena, where the elliptic equation is the most basic class of partial differential equations in elliptic differential equations. This equation governs a variety of equilibrium physical phenomena such as temperature distribution in solids, electrostatics, potential flow, ground water flow and many other fields of science and technology.

At present, there are many numerical methods to solve partial differential equations, such as boundary element method (BEM) and the finite element method (FEM), finite difference methods (FDM), spectral analysis and so on. One of the easiest to use is the finite difference method.

The steady heat equation is an importrant equation in heat transfer, which is widely used in a large number of enginering practice,Therefore, it is a key problem in the field of numerical computation to find a simple and quick numerical solution method. For these reasons, there are still a large number of researchers in this area in-depth study[1,2]. In this paper, we focus on the use of finite difference methods, a FDM scheme was propsed to calculate the steady heat conduction equation.

2. Hybrid Difference Scheme

Finite difference methods approximate the derivative of a function at a given point by a finite difference. A general discussion on finite difference methods for partial differential equations can be found, e.g. in [3,4].

For the 2D steady heat conduction equation in a rectangular domain,

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \qquad (1)$$

Suppose we chose rectangular area, the grid points are uniformly spaced,$\Delta x=\Delta y=h$ .Fig.1 shows a rectangular patterns of uniformly spaced grid samples.

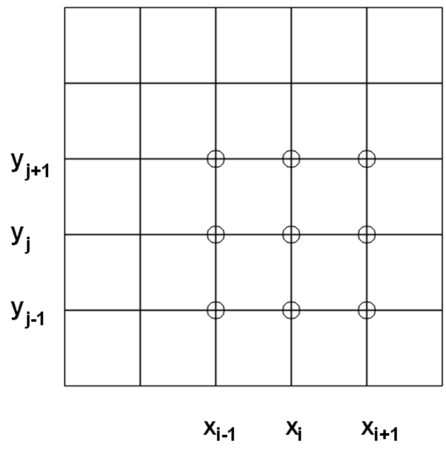

Fig.1 The grid used with Laplace's difference equation

In [5], a 5-point difference scheme of the above equation is given.This formula relates the value u(i,j) to four points u(i+1,j),u(i-1,j),u(i,j+1),u(i,j-1),as shown in Fig.2 (a)

$$u(i+1,j)+u(i-1,j)+u(i,j+1)+u(i,j-1)-4u(i,j)=0 \tag{2}$$

In [6], another 5-point difference scheme is given, This formula relates u(i,j) to four points u(i+1,j+1),u(i-1,j-1),u(i+1,j-1),u(i-1,j+1),as shown in Fig.2 (b).

$$u(i+1,j+1)+u(i-1,j+1)+u(i+1,j-1)+u(i-1,j-1)-4u(i,j)=0 \tag{3}$$

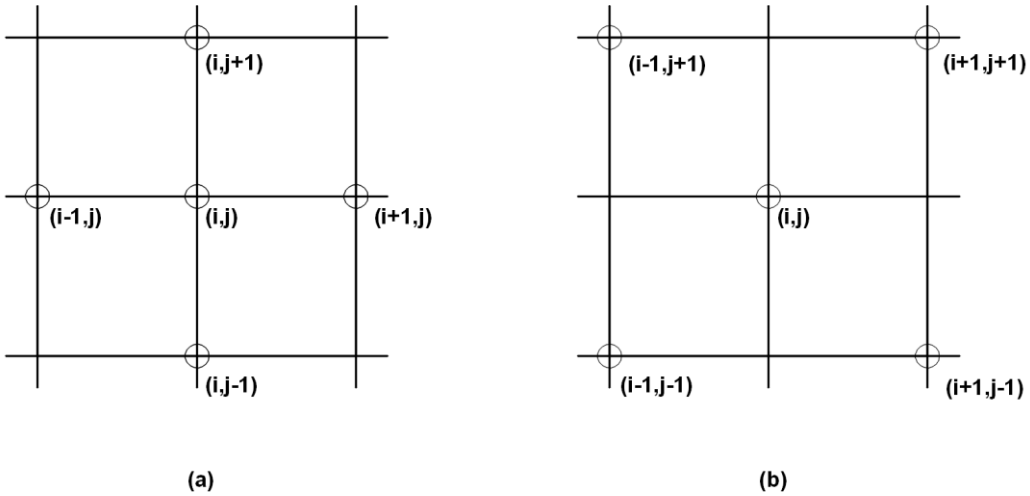

Fig.2 Finite difference stencil: five-point stencil

This paper is mainly for the 2D steady conduction equation, given a difference scheme. This scheme staggered using two 5-point schemes, so in this paper called hybrid scheme.the hybrid scheme relates the u(i,j) to eight points, see Fig.3.

In the actual operation of the first step using the 5-point format (a), the second step is to use the second 5-point (b), two 5-point scheme cross calculations. Compared to the 5-point schemes, hybrid scheme is a variant of the 9-point schemes , Which has order of accuracy $O(h^4)$.The method has the advantage of easier to implement of 5-point scheme, and also has the advantages of high precision of 9-point scheme.

In the actual numerical calculation, the C code of the hybrid scheme is as follows:

```
if (0==i%2)
t[i][j]=0.25*(t[i+1][j]+t[i-1][j]+t[i][j+1]+t[i][j-1]);
else
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t[i][j]=0.25*(t[i+1][j+1]+t[i+1][j-1]+t[i-1][j+1]+t[i-1][j-1]);

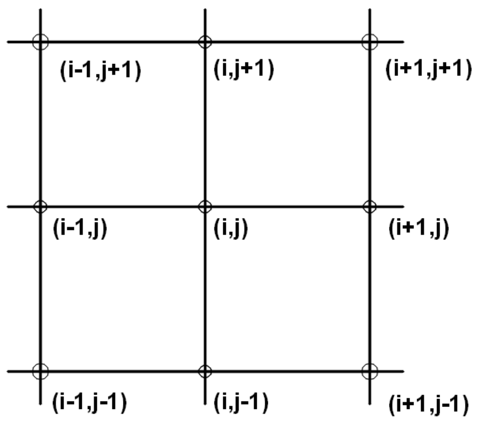

Fig.3 Hybrid difference stencil

3. Numerical Examples

In this section, In order to verify the computational speed and accuracy of the hybrid algorithm, we present two numerical examples that we have the exact solution to test the hybrid scheme [5].

3.1 Example1

We considering the Dirichlet problem is posed on a square domain as follows:

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \tag{4}$$

For u(x,y) defined within the square area of 0≤x≤1 and 0≤y≤1,given the boundary conditions:

$$\begin{cases} u(x,0) = 0, u(x,1) = x, 0 \le x \le 1 \\ u(0,y) = 0, u(1,y) = y, 0 \le y \le 1 \end{cases} \tag{5}$$

the exact solution of above equation is:

$$u(x,y) = xy \tag{6}$$

When the problem is solved on 6×6 grid and 12×12 grid , respectively ,with five point scheme,nine point and the hybrid scheme.The iteration converges result of these schemes is shown in Table 1.

Table 1 Iteration numbers, using three schemes with different grid numbers

	5-point	9-point	hybrid
Tolerance=10e-6, 6x6 grid	25	22	20
Tolerance =10e-9, 12x12 grid	179	159	144

3.2 Example 2

We consider the problem,

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \tag{7}$$

u(x,y) defined within the domain of 0≤x≤1 and 0≤y≤1,the boundary conditions as follow,

$$\begin{cases} u(x,0) = u(x,1) = u(0,y) = 0 \\ u(1,y) = \sin \pi y \end{cases} \tag{8}$$

the exact solution of above equation is:

$$u(x,y) = \frac{sh(\pi x)}{sh(\pi)} \sin(\pi y) \tag{9}$$

Solving the above equations by hybrid scheme,five points schemes ,nine points schemes .we get result in Table 2.

Table 2 Iteration numbers, using three schemes with different grid numbers

	5-point	9-point	hybrid
Tolerance=10e-6, 6x6 grid	20	18	16
Tolerance =10e-9, 12x12 grid	188	160	148

The algorithms were implemented in C using double precision arithmetic.

In Table 1 and Table 2, we compare the computational results of hybrid scheme , 5-point scheme and 9-point scheme in the steady conduction equation. It is apparent hybrid schemes is less than the other two schemes iteration numbers.As the grid grows,the advantage of the iteration numbers is more pronounced.

4. Conclusion

Two numerical examples which steady heat equation in 2-dimensional domain with Dirichlet boundary conditions are presented to see the performance of the hybrid FDM scheme. The numerical results show that the number of iterations is less than that of the 5-point and 9-point schemes when the differential equation is solved. The hybrid FDM scheme, for compute the steady heat conduction equations, is both simple and effective.

References

[1] Parag V.Patil, Dr.J.S.V.R.Krishna Prasad: Numrical sSlution for Two Dimensional Laplace Equation with Dirichlet Boundary Conditions,Journal of Mathematics,Vol.6(2013)NO.(4),p.66-75

[2] H.Bennour and M.S.Said: Numerical Solution of Poisson equation with Dirichlet Boundary Conditions, International Journal of Open Problems in Computer Science & Mathematics, Vol. 5(2012) NO.4 ,p.171-194

[3] J. C. Strikwerda: Finite Difference Schemes and Partial Differential Equations(Wads worth & Brooks/Cole, Pacific Groove, CA, 1989).p.311-325

[4] Leon lapidus ceorge F.Pinder : Numerical Solution of Partial Differential Equations Science and Engineering(John Wiley Sons ,CA,1999). p. 360-383

[5] John H. Mathews and Kurtis D. Fink: Numerical Methods Using Matlab,4th Ed(Pearson Education Asia Limited and Publishing House of Electronics Industry, China,2005). p.538-545

[6] C. Gong ,Z.L. Wang:MATLAB commonly used algorithm assembly(Publishing House of Electronics Industry, China, 2008), p.472-479. (In Chinese).