Endomorphism Types of Trapezoidal Graphs

Yanhua Song

School of School of Mathematics and Statistics, Henan University of Science and Tecnology, Luoyang, Henan, 471023, China.

458804756@qq.com

Abstract

For a graph, we have six families of endomorphisms: endomorphism, halfstrong endomorphism, locally strong endomorphism, quasi-strong endomorphism, strong endomorphism and automorphism. In order to describe the symmetry of a graph, U. Knauer define the endomorphism spectra and endomorphism types [see 1] of a graph. In this article, we give the endomorphism types of trapezoidal graphs.

Keywords

Graph; endomorphism type; endomorphism .

1. Introduction

Endomorphism monoid of graphs is a very important field of study, more and more people begin to explore the nature of endomorphism monoid. In recent years, more and more results about the endomorphism monoid graph have been given. In[2], Hou characterize the endomorphism types of generalized polygons.In [3], Fan explored the endomorphism spectra of bipartite graphs with diameter three and girth six. There are many excellent conclusions to be made [4],[5],[6]. In this paper ,we consider the endomorphism types of Trapezoidal Graphs .

2. Terminology.

Let X be a graph. Then X is called a trapezoidal graphs if $V(x)=\{x_1,x_2,\cdots x_n\}\cup\{y_1,y_2,\cdots y_n\}$, where x_i is adjacent to x_{i+1} for any $i\in\{1,2,\cdots n-1\}$, and y_i is adjacent to y_{i+1} for any $i\in\{1,2,\cdots n-1\}$, and x_i and y_i are andjacent in for any $i\in\{1,2,\cdots n\}$.

We consider graphs X with vertex set $V(X)$ and edge set $E(X)$. Let X and Y be two graphs, a mapping f from $V(X)$ to $V(Y)$ is called a endomorphism if f satisfy that $\{x,y\}\in E(X)$ implies $\{f(x),f(y)\}\in E(Y)$ .The endomorphism f is called a halfstrong endomorphism if $\{f(x),f(y)\}\in E(Y)$ implies that there exist $\bar{x}\in f^{-1}(f(x))$, $\bar{y}\in f^{-1}(f(y))$, such that $\{x,y\}\in E(X)$.

The endomorphism f is called a locally strong endomorphism,if $\{f(x),f(y)\}\in E(Y)$,implies that for every preimages $\bar{x}$ of $f(x)$,there exists a preimage $\bar{y}$ of $f(y)$ such that $\{\bar{x},\bar{y}\}\in E(X)$,and analogously for every preimage of $f(y)$ .The endomorphism f is called a quasi-strong endomorphism,if $\{f(x),f(y)\}\in E(Y)$ implies that exists a preimage $\bar{x}\in X$ of $f(x)$ which is abjacent to every preimage of $f(y)$ and analogously for every preimage of $f(y)$.The endomorphism f is called to be a strong endomorphism ,if $\{f(x),f(y)\}\in E(Y)$ implies that any preimage of $f(x)$ is abjacent to any preimage $f(y)$ .The endomorphism f is called to be a automorphism, if f is a bijective and f^{-1} is an endomorphism .

The sets of endomorphisms , half-strong endomorphisms ,locally strong endomorphisms , quasi-strong endomorphisms ,strong endomorphisms and automorphisms of X are respectively denoted by EndX, HEndX, LEndX, QEndX, SEndX, AutX. We have this sequence

$\mathrm{EndX} \supseteq \mathrm{HEndX} \supseteq \mathrm{LEndX} \supseteq \mathrm{QEndX} \supseteq \mathrm{SEndX} \supseteq \mathrm{AutX}$ With the sequence ,we associate the sequece of respective cardinalities by

$EndspecX = (|EndX|,|HEndX|,|LEndX|,|QEndX|,|SEndX|,|AutX|)$,and call the 6-tuple the endomorphism spectrum of X .We associate with the above sequece a 5-tuple $(s_1, s_2, s_3, s_4, s_5)$with $s_i \in \{0,1\}, i = 1,2,3,4,5$,where 1 stands for $\neq$ and 0 stands for $=$ at the respective position in the above sequence. The integer $\sum_{i=1}^{5} s_i 2^{i-1}$ is called the endomorphism type of X .

3. Main Result

We consider the Endomorphism Types of Trapezoidal Graphs .The main result is the following theorem.

If $n = 2$,the trapezoidal graph is a complete bipartite graph .the endomorphism types of generalized polygons have been given in []. In the following, we let $n \geq 3$.

Therem 2.1 Let $T_n(X)$ is a trapezoidal Graph .Then $Aut(X) \neq SEnd(X)$.

Proof. Take $f \in SEnd(X)$ and $f(a) = f(b)$. Then $N(a) = N(b)$.Apparently when $n \geq 3$,There are no two vertices with the same neighborhood in X .Therefore, any endomorphism f cannot put two vertices onto a vertices. So, $f \in Aut(X)$.

In conclusion , $Aut(X) \neq SEnd(X)$.

Therem 2.2 Let $T_n(X)$ is a trapezoidal Graph .Then $End(X) \neq HEnd(X)$

Proof. Let $X \in (Tn(X))$, Define mapping as follows:

$$f(x) = \begin{cases} x_1 & x = x_1 \\ x_2 & x = y_1, x_2 \\ x_3 & x = y_i, x_j, \quad \text{其中} i = 4,6,8\cdots, j = 3,5,7\cdots \\ x_4 & x = y_j, x_i, \quad \text{其中} i = 4,6,8\cdots, j = 3,5,7\cdots \end{cases}$$

$$g(x) = \begin{cases} x_1 & x = x_1 \\ x_2 & x = y_1, x_2 \\ y_2 & x = y_i, x_j \quad \text{其中} \mathrm{i} = 4,6,8\cdots, \ \mathrm{j} = 3,5,7\cdots \\ y_1 & x = y_j, x_i \quad \text{其中} \mathrm{i} = 4,6,8\cdots, \ \mathrm{j} = 3,5,7\cdots \end{cases}$$

Then, $f, g \in End(X)$, $x_1, y_1 \in V(I_f)$ and $\{x_1, y_1\} \in E(x)$ .The preimage of $f(x_1)$ is x_1, $f^{-1}(f(x_1)) = x_1$,the preimage of $f(y_1)$ is $\{y_3, y_5, y_7 \cdots\} \cup \{x_4, x_6, x_8 \cdots\}$, so , x_1 is not adjacent to y_1 .Therefore $f, g \notin HEnd(X)$.

In conclusion, $End(X) \neq HEnd(X)$ when $n \geq 3$.

Therem 2.3 Let $T_n(X)$ is a trapezoidal Graph .Then $HEnd(X) \neq LEnd(X)$

Proof. Let $T_n(X)$ be a trapezoidal Graph, Define the mapping as follows:

$$f = \begin{pmatrix} x_1 & y_1 & x_2 & y_2 & x_3 & y_3 & x_4 \cdots y_{n-1} & x_n & y_n \\ x_3 & x_2 & x_2 & x_3 & x_3 & x_4 & x_4 \cdots & x_n & x_n & x_{n-1} \end{pmatrix}$$

Apparently $\{x_3, x_4\} \in E(x)$ and $x_3, x_4 \in V(I_f)$, $f^{-1}(x_3) = \{x_1, y_2, x_3\}$, $f^{-1}(x_4) = \{y_3, x_4\}$ .Due to the $x_1 \in f^{-1}(x_3)$,but every preimages of x_4 is not adjacent to x_1. Then $f \notin LEnd(X)$. The image of f is a path with vertices $\{x_2, x_3, \cdots x_n\}$,it is an induced subgraph. Hence $f \in HEnd(X)$.

In conclusion, $HEnd(X) \neq LEnd(X)$ when $n \geq 3$.

Therem 2.4 Let $T_n(X)$ is a trapezoidal Graph.Then $LEnd(X) \neq QEnd(X)$

Proof.When $n = 3$,for any of the vertex $X \in (Tn(X))$, define mapping as follows:

$$f = \begin{pmatrix} x_1 & x_2 & x_3 & y_1 & y_2 & y_3 \\ x_1 & x_2 & x_1 & y_1 & y_2 & y_1 \end{pmatrix}$$

Absolutely, $\{x_1, y_1\} \in E(I_f)$, $f^{-1}(x_1) = \{x_1, x_3\}$, $f^{-1}(y_1) = \{y_1, y_3\}$ .For every preimages $x \in f^{-1}(x_1)$,there exists a preimage $y \in f^{-1}(y_1)$ such that $\{x, y\} \in E(x)$.It is routine verify that $f \in LEnd(X)$.And because of x_1 is not adjacent to y_3, y_1 is not adjacent to x_3.Then there is not preimage $x \in f^{-1}(x_1)$ such that x is adjacent to every vertices of $f^{-1}(x_1)$.It is routine to verify that $f \notin QEnd(X)$.so, $LEnd(X) \neq QEnd(X)$ when $n = 3$.Now we start to consider the case $n \geq 4$.

Divide vertex set of $T_n(X)$ into two parts, $A = \{x_1, y_2, x_3, y_4, \cdots\}$ and $B = \{y_1, x_2, y_3, x_4, \cdots\}$.and the vertices of A are not adjacent to vertices of B,Thus $T_n(X)$ is a bipartite graph .Define mapping as follows:

$$f(x) = \begin{cases} x_1 & x \in A \\ x_2 & x \in B \end{cases}$$

Obviously , for every preimages $x \in f^{-1}(x_1)$,there exists a preimage $y \in f^{-1}(x_2)$ such that $\{x, y\} \in E(x)$. analogously for every preimages of $f(x_2)$ .so, $f \in LEnd(X)$.

Now we prove $f \notin QEnd(X)$.If $f \in QEnd(X)$, there exists a preimage $x \in A = \{x_1, y_2, x_3, y_4, \cdots\}$ of $f(x_1)$,such that x is adjacent to every preimages of $f(x_2)$.In other words , x is adjacent to $y_1, x_2, y_3, x_4 \cdots$.But, obviously, that's not true. Then $f \notin QEnd(X)$.

In conclusion, $LEnd(X) \neq QEnd(X)$ when $n \geq 4$.

Therem 2.5 Let $T_n(X)$ be a trapezoidal Graph .Then $LqEnd(X) \neq SEnd(X)$

Proof.For any of the vertex $X \in (Tn(X))$, Define mapping as follows:

$$f = \begin{pmatrix} x_1 & y_1 & x_2 & y_2 & x_3 & y_3 & x_4 \cdots y_{n-1} & x_n & y_n \\ x_1 & x_2 & x_2 & x_3 & x_3 & x_4 & x_4 \cdots & x_n & x_n & y_n \end{pmatrix}$$

Apparently ,the image of homomorphism is a path. Take $\{x_i, x_{i+1}\} \in E(I_f)$,then $x_i \in f^{-1}(x_i)$, x_i is adjacent to x_{i+1}, y_i.In other words x_i is adjacent to every vertexes of $f^{-1}(x_{i+1})$.In the same way,for $y_i \in f^{-1}(x_{i+1})$, y_i is adjacent to x_i, y_{i-1}, in other words, y_i is adjacent to every vertexes of $f^{-1}(x_i)$.so $f \in QEnd(X)$,but , $f \notin SEnd(X)$. Theorem 1 has been proved.

In conclusion, $LqEnd(X) \neq SEnd(X)$ when $n \geq 3$.

Through the whole article, we can get that endomorphism types of trapezoidal graphs as follows:

$$EndotypeT_n(X) = \begin{cases} 0 & n = 2 \\ 0 & n \geq 3 \end{cases}$$

Acknowledgements

Natural Science Foundation.

References

[1] Böttcher M, Knauer U: Endomorphism spectra of graphs, Discrete Mathematics, Vol.109(1992) No.1-3, p.45-57.

[2] Hou H, Fan X, Luoy Y: Endomorphism Types of Generalized Polygons, Southeast Asian Bulletin of Mathematics, Vol. 33(2009) No. 3, p.433-441.

[3] Fan S: Endomorphism spectra of bipartite graphs with diameter three and girth six，Southeast Asian Bulletin of Mathematics, Vol. 25(2001) No.2, p.217–221.

[4] Fan S: Weakly Symmetric Graphs and Their Endomorphism Monoids, Southeast Asian Bulletin of Mathematics, Vol. 27(2003) No.3, p.439–444.

[5] Knauer U: Endomorphism types of trees, Words, Languages, and Combinatorics, World Scientific, Singapore, 1992, p.273-287.

[6] Wilkeit E: Graphs with a regular endomorphism monoid, Archiv der Mathematik, Vol. 66(1996) No.4, p.344–352.