Max-min Coordinates: a New Way to Generalize Barycentric Coordinates

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Abstract

Wehave solved the constrained optimization problem $\max\min\{\lambda_1(\boldsymbol{v}),\ldots,\lambda_n(\boldsymbol{v})\}$, **$\forall \boldsymbol{v} \in \Omega$, subject to the affine combination conditions $\sum_{i=1}^{n}\lambda_i(\boldsymbol{v}) = 1$ and $\sum_{i=1}^{n}\lambda_i(\boldsymbol{v})\boldsymbol{v}_i = \boldsymbol{v}$, where Ω is an arbitrary polytope with vertices $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$. The resulting coefficient functions λ_i, $i = 1,2,\cdots,n$ are unique and turn out to be both analytically elegant and geometrically intuitive. They also satisfy all the desired properties of generalized barycentric coordinates. We name these functions "max-min" coordinates and provide an efficient algorithm to compute them. Throughout this paper, we confine our discussions to the planar case, but all the results can easily be extended to higher dimensions.**

Keywords

Barycentric coordinates, convex polygon, convex optimization, max-min.

1. Introduction

1.1 Barycentric coordinates

Barycentric coordinates for simplices are commonly used in various computational sciences. They provide a natural and often the unique way of parameterizing the inside points, or weighting all kinds of data sampled on the vertices. The need to generalize these coordinates to arbitrary convex polytopes has long been the interest in many applications such as boundary interpolation, texture mapping and free-form deformation.

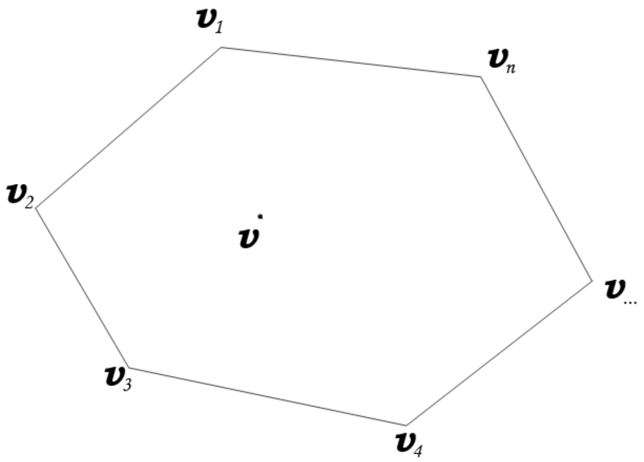


Figure 1. A convex n-gon with labeled vertices.

The algebraic definition of this coordinates system is similar for all dimensions, so we will confine our discussion to the 2D case. Let Ω be a planar convex n-gon, as shown in Figure 1, where bold $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, ..., $\boldsymbol{v}_n$ are used to denote all the vertices. For convenience, we will henceforth identify all points $p \in \Omega$ with points $p \in \mathrm{R}^3$, whose third coordinates are all one.

For any point $\boldsymbol{v} \in \Omega$, a set of *non-negative* real numbers (λ_1, λ_2, ... , λ_n) are called *barycentric coordinates* of v with respect to Ω, if

For any point $v \in \Omega$, a set of non-negative real numbers ($\lambda_1, \lambda_2, \ldots, \lambda_n$) are called barycentric coordinates of v with respect to Ω, if

$$\sum_{i=1}^{n} \lambda_i \boldsymbol{v}_i = \boldsymbol{v} \tag{1}$$

And

$$\sum_{i=1}^{n} \lambda_i = 1 \tag{2}$$

Since all the points lie on the same plane and we already add a third component "1" to each of them, the property (2) can be deduced from (1) and is thus redundant in our discussion.

The fact that Ω is convex guarantees that the above defined barycentric coordinates always exist for any $\boldsymbol{v} \in \Omega$. They also are unique if $\boldsymbol{v}$ lies on the boundary of Ω. We provide a short proof here, which again shows the convenience of adding a third component. Suppose $\boldsymbol{v}$ lies on the edge v_1v_2, and $\boldsymbol{v} = \lambda_1\boldsymbol{v}_1 + \lambda_2\boldsymbol{v}_2$. Assume there is another tuple of barycentric coordinates $(\mu_1,\mu_2,\ldots,\mu_n)$ other than $(\lambda_1,\lambda_2,0,\ldots,0)$, then

$$\boldsymbol{v} = \lambda_1\boldsymbol{v}_1 + \lambda_2\boldsymbol{v}_2 = \mu_1\boldsymbol{v}_1 + \mu_2\boldsymbol{v}_2 + \cdots + \mu_n\boldsymbol{v}_n. \tag{3}$$

Thus

$$|\boldsymbol{v}_1\boldsymbol{v}_2\boldsymbol{v}| = 0 = \mu_3|\boldsymbol{v}_1\boldsymbol{v}_2\boldsymbol{v}_3| + \cdots + \mu_n|\boldsymbol{v}_1\boldsymbol{v}_2\boldsymbol{v}_n| \tag{4}$$

Where, for example, $|\boldsymbol{v}_1\boldsymbol{v}_2\boldsymbol{v}_3|$ is the determinant and represents the signed area of the triangle $v_1v_2v_3$. All the determinants $|\boldsymbol{v}_1\boldsymbol{v}_2\boldsymbol{v}_k|(k = 3,\ldots,n)$ share the same sign because all the $\boldsymbol{v}_k(k = 3,\ldots,n)$ lie on the same side of the edge v_1v_2. Thus all the $\mu_k(k = 3,\ldots,n)$ equal zero considering their non-negativity, and then $\lambda_1 = \mu_1$, $\lambda_2 = \mu_2$ follow.

For $\boldsymbol{v}$ that lies inside Ω and not on the boundary, there are infinitely many ways of choosing its barycentric coordinates if $n \geq 4$. The correspondences between points in Ω and their barycentric coordinates are called barycentric coordinates functions. We will denote these functions by b_i, $i = 1,\ldots,n$, and rewrite property (1) as

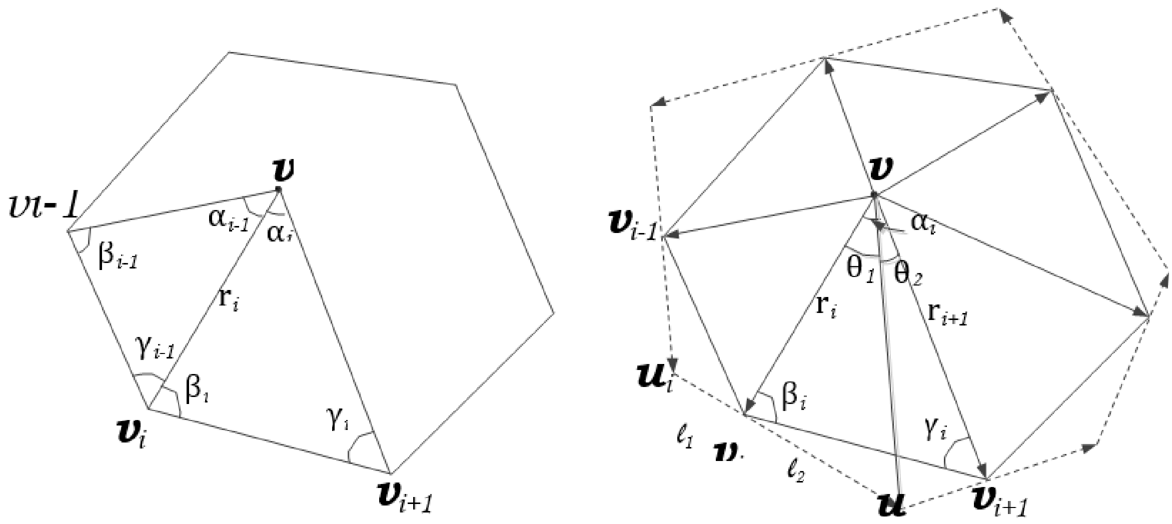
$$\textstyle\sum_{i=1}^{n} b_i(\boldsymbol{v})\boldsymbol{v}_i = \boldsymbol{v}, \quad \forall v \in \Omega. \tag{5}$$

Although there are infinitely many ways of choosing b_i, continuous functions are preferred in various applications and smooth functions, sometimes even rational functions are required [7].

1.2 A note on previous work

In fact, there are infinitely many ways of constructing barycentric coordinates, but three main constructions are widely used, namely the *Wachspress coordinates* proposed by Wachspress[4], further investigated and generalized by others[6][3][7], the *Sibson coordinates*[5], and the *mean value coordinates*[1]. Floater etc[2] actually give a general construction method for all generalized barycentric coordinates, and categorize both Wachspress coordinates and mean value coordinates into a group called *three-point coordinates*. Here we provide an alternative geometric construction that is both intuitive and general enough to coincide with Floater's general construction.

As is shown in Figure 2(b), the dashed line polygon (a loop) is formed by vectors that are perpendicular to all the radial vectors. For example, you can rotate vector vv_i counterclockwise and then rescale it to get u_iu_{i+1}. According to equation (5), we know that $\sum_{i=1}^{n} b_i(\boldsymbol{v})(\boldsymbol{v}_i - \boldsymbol{v}) = 0$, which means barycentric coordinates are just the weights that balance the radial vectors to sum up to zero. Notice that the dashed vectors already sum up to zero because they form a loop, and the rotation transformation doesn't change this. So barycentric coordinates can just be the ratios used in the rescaling process.



(a) (b)

Figure 2: (a) The non-normalized weight ω. Wachspress: $\omega_i(\boldsymbol{v}) = \frac{2}{r_i^2}(\cot\gamma_{i-1} + \cot\beta_i)$; Mean value: $\omega_i(\boldsymbol{v}) = \frac{2}{r_i^2}(\tan\frac{\alpha_{i-1}}{2} + \tan\frac{\alpha_i}{2})$. (b) A general geometric construction

By using angles θ_1, θ_2 as shown in Figure 2(b), it is not difficult to get the length of the vector $v_i u_{i+1}$, denoted by l_2, and likewise the length of the vector $u_i v_i$, denoted by l_1. We get

$$l_2 = \frac{r_{i+1} - r_i \cos\alpha_i}{\sin\alpha_i}, l_1 = \frac{r_{i-1} - r_i \cos\alpha_{i-1}}{\sin\alpha_{i-1}}$$

Since all kinds of loops can be formed by just changing the length of the radial vectors, say, r_i to arbitrary a_i ($i = 1 \cdots n$, noting that this only changes the skeleton of the dashed virtual loop, not the original n-gon). So the edge length of the resulting dashed loop is

$$l_1 + l_2 = \frac{a_{i+1} - a_i \cos\alpha_i}{\sin\alpha_i} + \frac{a_{i-1} - a_i \cos\alpha_{i-1}}{\sin\alpha_{i-1}}$$

And the edge length of the original n-gon is r_i. Thus

$$\frac{l_1 + l_2}{r_i} = \frac{1}{r_i}\left(\frac{a_{i+1} - a_i \cos\alpha_i}{\sin\alpha_i} + \frac{a_{i-1} - a_i \cos\alpha_{i-1}}{\sin\alpha_{i-1}}\right)$$

Which is just the general construction of Floater’s in[2].

There are also other ways to generalize barycentric coordinates. For example, the maximum entropy coordinates [9] are constructed from the maximum entropy principle but difficult to compute. The affine generalized barycentric coordinates [10] are similar to our coordinates but with different norms.

2. The max-min coordinates

We now propose a completely new way of generalizing barycentric coordinates. Denote by S the set of all possible tuples of barycentric functions, whose element can be written as $F = (f_1, ..., f_n)$. The thought that the “barycenter” should have the coordinates $(\frac{1}{n}, \cdots, \frac{1}{n})$, gave the author the idea that we could pick from infinitely many tuples of barycentric functions the one whose components are more “equal” to each other. Hence the following definition:

Definition 2.1. Using the same notations as that of Figure 1. *An element* $B = (b_1, \ldots, b_n) \in S$ is called max-min if

$$\min\{b_1(\boldsymbol{v}),\dots,b_n(\boldsymbol{v})\} = \max_{\mathrm{F}=(f_1,\cdots,f_n)\in S} \min\{f_1(\boldsymbol{v}),\dots,f_n(\boldsymbol{v})\},\ \forall\ \boldsymbol{v}\in\Omega$$

The idea is that we want the smallest component to be as large as possible. To justify this definition, we have to prove that the max-min barycentric coordinates functions always exist and are unique.

Theorem 2.2. *For convex n-gon* Ω, *the above defined* max-min *barycentric coordinates functions* ($b_1,\dots,b_n$) *always exist.*

Proof. First, we describe a procedure we called *coordinates convergence*, which modifies an arbitrarily chosen tuple of barycentric coordinates functions to make it more "max-min". Let $F = (f_1,\dots,f_n)$ be the starting tuple where $n \ge 4$ and for a point $\boldsymbol{v} \in \Omega$, $f_i(\boldsymbol{v}) = \lambda_i$, $i = 1,\dots,n$, then $\boldsymbol{v} = \lambda_1\boldsymbol{v}_1+\cdots+\lambda_n\boldsymbol{v}_n$. Without loss of generality, let $\lambda_1 \le \lambda_k$, $k = 2,\dots,n$. If there is already an equality in these constraints, say $\lambda_1 = \lambda_2$, then combine the two terms to make

$$\mathrm{v} = \lambda_1(\mathrm{v}_1 + \mathrm{v}_2) + \lambda_3\mathrm{v}_3 + \cdots + \lambda_n\mathrm{v}_n \tag{6}$$

Otherwise, express $\boldsymbol{v}_1$ as $\boldsymbol{v}_1 = \mu_2\boldsymbol{v}_2+\mu_3\boldsymbol{v}_3+\cdots+\mu_n\boldsymbol{v}_n$, where $\mu_2+\cdots+\mu_n = 1$. Let

$$\Delta\lambda = \min_{\substack{i=2..n\\ 1+\mu_i>0}} \left\{\frac{\lambda_i-\lambda_1}{1+\mu_i}\right\} \xlongequal[\text{of generality}]{\text{without loss}} \left\{\frac{\lambda_2-\lambda_1}{1+\mu_2}\right\} \tag{7}$$

Then clearly $\Delta\lambda > 0$ and we get

$$\mathrm{v} = (\lambda_1 + \Delta\lambda)(\mathrm{v}_1 + \mathrm{v}_2) + (\lambda_3 - \mu_3\Delta\lambda)\mathrm{v}_3 + \cdots + (\lambda_n - \mu_n\Delta\lambda)\mathrm{v}_n \tag{8}$$

An expression just like that of (6). Also, (7) indicates that $\lambda_1+\Delta\lambda \le \lambda_i-\mu_i\Delta\lambda$, for $i = 3,\dots,n$. If still $n-1 \ge 4$, then perform the same procedure to expression (6) or (8). Continue with this $n - 3$ times, we will reach the final modified expression looking like this:

$$\boldsymbol{v} = \lambda_1'(\boldsymbol{v}_1+\boldsymbol{v}_2+\cdots+\boldsymbol{v}_{n-2}) + \lambda_{n-1}'\boldsymbol{v}_{n-1} + \lambda_n'\boldsymbol{v}_n \tag{9}$$

where $\lambda_1' \le \lambda_{n-1}', \lambda_1' \le \lambda_n'$ and $\lambda_1' \ge \lambda_1$. There are exactly $\binom{n}{3}$ modified expressions that all others can convergent into. So the values of the max-min barycentric coordinates functions at point v must be the coefficients of one of the modified expressions, thus exist. □

According to this proof, the max-min barycentric coordinates of any points $\boldsymbol{v} \in \Omega$ must take the form of $(\lambda,\dots,\lambda_i,\dots,\lambda_j,\dots,\lambda)$, where $i < j$, $\lambda_i \ge \lambda$, $\lambda_j \ge \lambda$ and all other components equal λ.

Theorem 2.3. *For convex n-gon* Ω, *the* max-min *barycentric coordinates functions* ($b_1,\dots,b_n$) *are unique.*

Proof. It is enough to prove that for any point $v \in \Omega$, the coordinates form $(\lambda,\dots,\lambda_i,\dots,\lambda_j,\dots,\lambda)$ is unique. First, we show that for this form to be max-min, it must be that

case 1: $\lambda_i = \lambda$ or $\lambda_j = \lambda$; Or else,

case 2: if $\lambda_i > \lambda$ and $\lambda_j > \lambda$, then either $j = i + 1$, or $i = 1$ at the same time $j = n$, which means, v_i and v_j are adjacent points.

Let c be the barycentric center of Ω, which means

$$\boldsymbol{c} = \frac{1}{n}(\boldsymbol{v}_1+\boldsymbol{v}_2+\cdots+\boldsymbol{v}_n) \tag{10}$$

Then

$$\begin{aligned}\boldsymbol{v} &= \lambda \boldsymbol{v}_1 + \cdots + \lambda_i \boldsymbol{v}_i + \cdots + \lambda_j \boldsymbol{v}_j + \cdots + \lambda \boldsymbol{v}_n \\ &= (\lambda_i - \lambda)\boldsymbol{v}_i + (\lambda_j - \lambda)\boldsymbol{v}_j + n\lambda \boldsymbol{c} \end{aligned} \tag{11}$$

We can solve this to get

$$\lambda_i - \lambda = \frac{|\boldsymbol{c}\boldsymbol{v}\boldsymbol{v}_j|}{|\boldsymbol{c}\boldsymbol{v}_i\boldsymbol{v}_j|} \geq 0, \tag{12}$$

$$\lambda_j - \lambda = -\frac{|\boldsymbol{c}\boldsymbol{v}\boldsymbol{v}_i|}{|\boldsymbol{c}\boldsymbol{v}_i\boldsymbol{v}_j|} \geq 0, \tag{13}$$

$$n\lambda = \frac{|\boldsymbol{v}\boldsymbol{v}_i\boldsymbol{v}_j|}{|\boldsymbol{c}\boldsymbol{v}_i\boldsymbol{v}_j|} \geq 0, \tag{14}$$

If $\lambda_i > \lambda$ and $\lambda_j > \lambda$, we show that points $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ are adjacent: If not, then there must be a point $\boldsymbol{v}_k$ between them which can be expressed as

$$\boldsymbol{v}_k - \boldsymbol{c} = \mu_1(\boldsymbol{v}_i - \boldsymbol{c}) + \mu_2(\boldsymbol{v}_j - \boldsymbol{c}),\ \mu_1 + \mu_2 > 1,\ \mu_1 > 0,\ \mu_2 > 0. \tag{15}$$

We then have $\boldsymbol{v}_k = \mu_1\boldsymbol{v}_i + \mu_2\boldsymbol{v}_j + (1 - \mu_1 - \mu_2)\boldsymbol{c}$, and could get two new coordinates form $(\lambda',\ldots,\lambda'_i,\ldots,\lambda'_k,\ldots,\lambda')$ and $(\lambda'',\ldots,\lambda''_j,\ldots,\lambda''_k,\ldots,\lambda'')$, by solving two similar vector equations as that of (11), where (i,j) is replaced by (i,k) and (j,k) respectively. We have

$$\lambda'_i - \lambda' = \frac{|\boldsymbol{c}\boldsymbol{v}\boldsymbol{v}_k|}{|\boldsymbol{c}\boldsymbol{v}_i\boldsymbol{v}_k|} = \frac{\mu_1|\boldsymbol{c}\boldsymbol{v}\boldsymbol{v}_i| + \mu_2|\boldsymbol{c}\boldsymbol{v}\boldsymbol{v}_j|}{\mu_2|\boldsymbol{c}\boldsymbol{v}_i\boldsymbol{v}_j|}, \tag{16}$$

$$\lambda'_k - \lambda' = -\frac{|\boldsymbol{c}\boldsymbol{v}\boldsymbol{v}_i|}{|\boldsymbol{c}\boldsymbol{v}_i\boldsymbol{v}_k|} = -\frac{|\boldsymbol{c}\boldsymbol{v}\boldsymbol{v}_i|}{\mu_2|\boldsymbol{c}\boldsymbol{v}_i\boldsymbol{v}_j|} > 0, \tag{17}$$

$$\begin{aligned} n\lambda' &= \frac{|\boldsymbol{v}\boldsymbol{v}_i\boldsymbol{v}_k|}{|\boldsymbol{c}\boldsymbol{v}_i\boldsymbol{v}_k|} = \frac{\mu_2|\boldsymbol{v}\boldsymbol{v}_i\boldsymbol{v}_j| + (1-\mu_1-\mu_2)|\boldsymbol{c}\boldsymbol{v}\boldsymbol{v}_i|}{\mu_2|\boldsymbol{c}\boldsymbol{v}_i\boldsymbol{v}_j|} \\ &= \frac{|\boldsymbol{v}\boldsymbol{v}_i\boldsymbol{v}_j|}{|\boldsymbol{c}\boldsymbol{v}_i\boldsymbol{v}_j|} + \frac{1-\mu_1-\mu_2}{\mu_2}\frac{|\boldsymbol{c}\boldsymbol{v}\boldsymbol{v}_i|}{|\boldsymbol{c}\boldsymbol{v}_i\boldsymbol{v}_j|} > n\lambda, \end{aligned} \tag{18}$$

and

$$\lambda''_k - \lambda'' = \frac{|\boldsymbol{c}\boldsymbol{v}\boldsymbol{v}_j|}{|\boldsymbol{c}\boldsymbol{v}_k\boldsymbol{v}_j|} = \frac{|\boldsymbol{c}\boldsymbol{v}\boldsymbol{v}_j|}{\mu_1|\boldsymbol{c}\boldsymbol{v}_i\boldsymbol{v}_j|} > 0, \tag{19}$$

$$\lambda''_j - \lambda'' = -\frac{|\boldsymbol{c}\boldsymbol{v}\boldsymbol{v}_k|}{|\boldsymbol{c}\boldsymbol{v}_k\boldsymbol{v}_j|} = -\frac{\mu_1|\boldsymbol{c}\boldsymbol{v}\boldsymbol{v}_i| + \mu_2|\boldsymbol{c}\boldsymbol{v}\boldsymbol{v}_j|}{\mu_1|\boldsymbol{c}\boldsymbol{v}_i\boldsymbol{v}_j|}, \tag{20}$$

$$\begin{aligned} n\lambda'' &= \frac{|\boldsymbol{v}\boldsymbol{v}_k\boldsymbol{v}_j|}{|\boldsymbol{c}\boldsymbol{v}_k\boldsymbol{v}_j|} = \frac{\mu_1|\boldsymbol{v}\boldsymbol{v}_i\boldsymbol{v}_j| + (\mu_1+\mu_2-1)|\boldsymbol{c}\boldsymbol{v}\boldsymbol{v}_j|}{\mu_1|\boldsymbol{c}\boldsymbol{v}_i\boldsymbol{v}_j|} \\ &= \frac{|\boldsymbol{v}\boldsymbol{v}_i\boldsymbol{v}_j|}{|\boldsymbol{c}\boldsymbol{v}_i\boldsymbol{v}_j|} + \frac{\mu_1+\mu_2-1}{\mu_1}\frac{|\boldsymbol{c}\boldsymbol{v}\boldsymbol{v}_j|}{|\boldsymbol{c}\boldsymbol{v}_i\boldsymbol{v}_j|} > n\lambda. \end{aligned} \tag{21}$$

Notice that both λ' and λ'' are greater than λ, and either $\lambda'_i - \lambda' \geq 0$ or $\lambda''_j - \lambda'' \geq 0$. This means, at least one of the two new coordinates form is more max-min than $(\lambda,\ldots,\lambda_i,\ldots,\lambda_j,\ldots,\lambda)$, which is a contradiction. Thus $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ must be adjacent points.

Now we can go back to the proof of uniqueness. Assume there is another max-min coordinates form $(\lambda,\ldots,\lambda'_i,\ldots,\lambda'_j,\ldots,\lambda)$, which also falls into the above two cases. Then

$$\begin{aligned} \boldsymbol{v} &= \lambda \boldsymbol{v}_1 + \cdots + \lambda_i \boldsymbol{v}_i + \cdots + \lambda_j \boldsymbol{v}_j + \cdots + \lambda \boldsymbol{v}_n \\ &= \lambda \boldsymbol{v}_1 + \cdots + \lambda'_i \boldsymbol{v}_i + \cdots + \lambda'_j v_j + \cdots + \lambda \boldsymbol{v}_n \end{aligned} \tag{22}$$

Which means

$$(\lambda_i - \lambda)\boldsymbol{v}_i + (\lambda_j - \lambda)\boldsymbol{v}_j = (\lambda'_i - \lambda)\boldsymbol{v}'_i + (\lambda'_j - \lambda)\boldsymbol{v}'_j \tag{23}$$

If both forms fall into **case** 1, then without loss of generality, let $\lambda_j = \lambda$, $\lambda'_j = \lambda$. Equation (23) then becomes $(\lambda_i - \lambda)\boldsymbol{v}_i = (\lambda'_i - \lambda)\boldsymbol{v}'_i$. This is possible only if $i = i'$, which means the two forms are the same.

On the other hand, if at least one of the two forms (for example the first one) falls into **case** 2, equation (23) can be normalized to:

$$\frac{\lambda_i - \lambda}{1 - n\lambda}\boldsymbol{v}_i + \frac{\lambda_j - \lambda}{1 - n\lambda}\boldsymbol{v}_j = \frac{\lambda_{i'} - \lambda}{1 - n\lambda}\boldsymbol{v}_{i'} + \frac{\lambda_{j'} - \lambda}{1 - n\lambda}\boldsymbol{v}_{j'} \tag{24}$$

Considering the fact that $(\lambda_i-\lambda)+(\lambda_j-\lambda) = (\lambda'_i-\lambda)+(\lambda'_j-\lambda) = 1-n\lambda$. Since $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ are adjacent points, the left part of equation (24) can be viewed as a boundary point of Ω, lying on edge $v_i v_j$. We already proved that the boundary points of Ω have unique non-negative coordinates, thus in equation (24), $i = i'$, $j = j'$. The uniqueness of the max-min coordinates form for v follows, and so does the conclusion of this theorem. □

This proof is also a way of calculating the max-min coordinates. We have already implemented in C/C++ the above algorithms for finding these coordinates.

3. Geometric interpretation of the max-min coordinates

As it turns out, the max-min coordinates have a very clear geometric interpretation. They are the simplest kind of barycentric coordinates, the triangulation coordinates, where the triangulations are determined by barycenters. See the pentagon example in Figure 3, where c is the barycenter. First determine which sub-triangle the point v falls in (in this case $4cv_{i-1}v_i$), then

$$\begin{aligned} b_{i-1}(\boldsymbol{v}) &= \frac{|\boldsymbol{v}\boldsymbol{v}_{i-1}\boldsymbol{v}_i| + 5|\boldsymbol{c}\boldsymbol{v}\boldsymbol{v}_i|}{5|\boldsymbol{c}\boldsymbol{v}_{i-1}\boldsymbol{v}_i|}, \\ b_i(\boldsymbol{v}) &= \frac{|\boldsymbol{v}\boldsymbol{v}_{i-1}\boldsymbol{v}_i| + 5|\boldsymbol{c}\boldsymbol{v}_{i-1}\boldsymbol{v}|}{5|\boldsymbol{c}\boldsymbol{v}_{i-1}\boldsymbol{v}_i|}, \\ b_{j(j\neq i-1,i)}(\boldsymbol{v}) &= \frac{|\boldsymbol{v}\boldsymbol{v}_{i-1}\boldsymbol{v}_i|}{5|\boldsymbol{c}\boldsymbol{v}_{i-1}\boldsymbol{v}_i|}, \end{aligned} \tag{25}$$

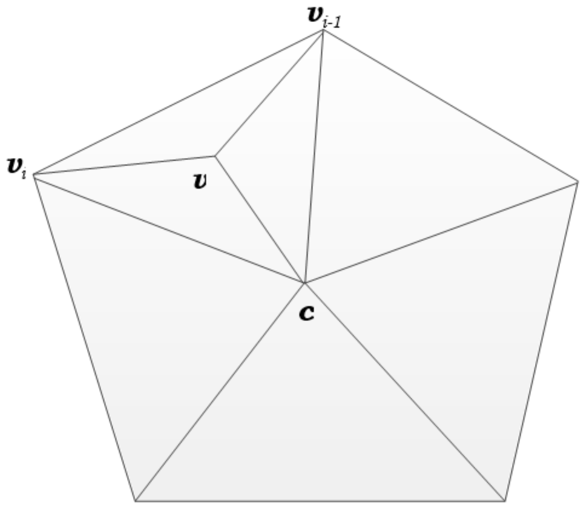


Figure 3. A pentagon example, where c is the barycenter

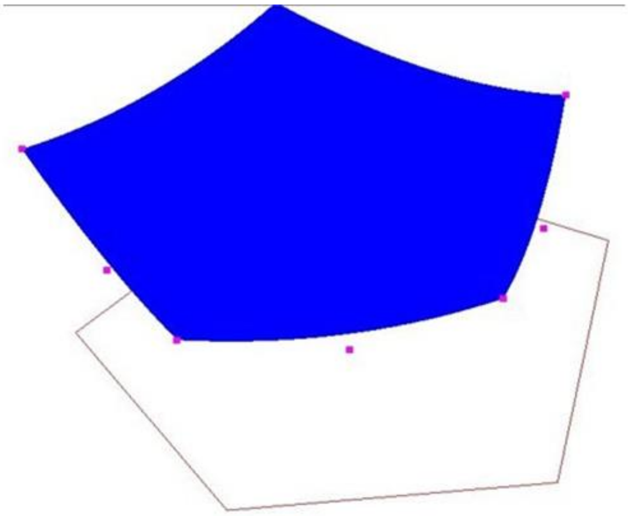


Figure 4. A Quadratic Interpolation with Control Points

4. C++ implementation of the max-min coordinates

Finally, we supply the main C++ function for computing the max-min coordinates.

```
double* CbcView::GetBaryc(double p[])
{
int flag1 = 0; int flag2 = 0; double coordiff1 = 0.0; double coordiff2 = 0.0; double mindiff = 1.0; for (int i = 0; i < numvert-1; i++)
{
for (int j = i + 1; j < numvert; j++)
{ double tempdenom = Det2(VectSub(vert[i], vert[numvert]), VectSub(vert[j], vert[numvert])); if (tempdenom != 0)
{ double tempnumer1 = Det2(VectSub(p, vert[numvert]), VectSub(vert[j], vert[numvert])); double tempdiff1 = tempnumer1 / tempdenom; if (tempdiff1 >= 0)
{ double tempnumer2 = Det2(VectSub(p, vert[numvert]), VectSub(vert[i], vert[numvert])); double tempdiff2 = -tempnumer2 / tempdenom; if (tempdiff2 >= 0)
{ if (tempdiff1 + tempdiff2 <= mindiff)
{
coordiff1 = tempdiff1; coordiff2 = tempdiff2; mindiff = coordiff1 + coordiff2; flag1 = i; flag2 = j;
}
}
}
}
} } double* bcp = new double[numvert]; for (int i = 0; i < numvert; i++)
{ bcp[i] = (1 - mindiff) / numvert;
} bcp[flag1] += coordiff1; bcp[flag2] += coordiff2; return bcp;
}
```

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