Existence of Periodic Loop Wave Solutions of Schrödinger Equation with Distributed Delay

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Abstract

In this paper we discuss existence problem of travelling wave solutions of a class of nonlinear Schrödinger equation with time delay. When the parameter is sufficiently small we establish results of the existence of periodic loop solutions of the nonlinear Schrödinger equation with time delay by the geometric singular perturbation theory and the Melnikov function method.

Keywords

Nonlinear Schrödinger equation, periodic loop wave solutions, geometric singular perturbation theory, Melnikov function, distributed delay.

1. Introduction

The nonlinear Schrödinger (NLS) equation which describes the law of the state of microscopic particles evoluting with time is widely applied in many fields, including nonlinear optics, atom、solid state physics, nuclear physics, chemistry and other fields [1-6]. This equation is completely integrable and its solitons, especially traveling wave solutions has been ongoing to investigate for several years [7-14].

The numerical analysis the effect of time delay on the solution of the NLS equation in [15-16]. Yang et al [17, 18] discuss the NLS equation with delay term that has much actual significance. Zhao and Ge [19] investigate the NLS equation with distributed delay and give the conditions that assure existence of the solitary wave and periodic solutions. We investigate existence of kink and anti-kink wave solutions in the distributed delay equation in [22]. We'll continue to study the NLS distributed delay equation and discuss existence of periodic loop wave solutions.

We'll consider the following NLS equation with distributed delay,

$$iU_t + U_{xx} - f * U|U|^2 - \tau U(|U|^2)_x = 0,\ -\infty < t < +\infty,\ -\infty < x < +\infty, \tag{1.1}$$

where $\tau > 0$ is time delay, $\tau\, U(|U|^2)_x$ means the nonlinear response delay term and $\tau = \int_0^{+\infty} tf(t)dt$, the convolution $f * U$ is defined by

$$(f * U)(x,t) = \int_{-\infty}^{t} f(t-s)U(x,s)ds, \tag{1.2}$$

and the kernel $f:[0,+\infty) \to [0,+\infty)$, that satisfies: $f(t) \ge 0$ for all $t \ge 0$ and $\int_0^{+\infty} f(t)dt = 1,\ tf(t) \in L^1((0,+\infty), R)$.

If $\tau = 0$ and $f(t) = \delta(t)$, where δ denotes Dirac δ function, Eq. (1.1) turns to the corresponding undelayed and undisturbed NLS equation:

$$iU_t + U_{xx} - U|U|^2 = 0. \tag{1.3}$$

In addition, if the different delay kernels were chosen, then the different types equations can be derived from Eq. (1.3) . For example, when we take the kernel to be $f(t)=\delta(t)$, then Eq. (1.1) becomes the corresponding original NLS equation: $iU_t+U_{xx}-U|U|^2-\tau U(|U|^2)_x=0$.

Gamma distribution delay kernel is often used $f(t)=\dfrac{\lambda^n t^{n-1}e^{-\lambda t}}{(n-1)!}$, $n=1,2,\ldots$, where $\lambda>0$ is a constant, n is a integer, with the average delay $\tau=n/\lambda>0$.

Two special cases $f(t)=\dfrac{1}{\tau}e^{-t/\tau}$ $(n=1)$ and $f(t)=\dfrac{t}{\tau^2}e^{-t/\tau}$ $(n=2)$ are called the weak generic kernel and the strong generic kernel, respectively.

In this paper, the distributed delay kernel $f(t)$ of Eq. (1.1) has the following form

$$f(t)=\frac{t}{\tau^2}e^{-t/\tau-iwt}, \tag{1.4}$$

where the parameter $w>0$.

The remaining parts are organized as follows. In Section 2, some preliminary theory and discussion are introduced. The periodic loop orbits of traveling waves for the non-delay equation (1.3) are given. In Section 3, we transform Eq. (1.1) with the strong generic kernel into a non-delay four-dimensional ordinary differential system. When the parameter τ is sufficiently small, he four-dimensional ordinary differential system is reduced to the two-dimensional system by the singular perturbation theory. We'll prove that there exists the periodic loop wave solutions of system (1.1) with the Melnikov function method.

2. Preliminaries

With traveling wave transformation, $U(x,t)=\varphi(\xi)e^{i\theta}$, $\xi=x-ct$, $\theta=ax-wt$, and $c>0$, where φ is real valued function and represents the amplitude of the traveling wave with wave number $a>0$ and frequency $w>0$.

Now substituting $U(x,t)=\varphi(\xi)e^{i\theta}=\varphi(x-ct)e^{i(ax-wt)}$ into the non-delay Eq. (1.3), one gets

$$\begin{aligned} &w\varphi+\varphi''-a^2\varphi+\varphi^3=0,\\ &-c\varphi'+2a\varphi'=0, \end{aligned} \tag{2.1}$$

where $'$denotes the derivative with respect to the variable ξ.

Let $a=c/2$ and $\mu=w-c^2/4$, Eq. (2.1) becomes

$$\varphi''=-\mu\varphi+\varphi^3. \tag{2.2}$$

Taking $u=\varphi/\sqrt{\mu}$ and $z=\sqrt{\mu}\xi$ to Eq. (2.2), one gets

$$\ddot{u}=-u+u^3, \tag{2.3}$$

where $\cdot$ denotes the derivative with respect to the variable z.

Namely

$$\begin{aligned} &\dot{u}=v,\\ &\dot{v}=-u+u^3. \end{aligned} \tag{2.4}$$

Lemma 2.1 In the (u,v) phase plane, Eq. (2.4) has a periodic loop orbit around the center $(0,0)$, so the periodic wave solution of non-delay equation (1.3) exists.

Proof. It's easy to see that Eq. (2.4) has three critical points $(0,0)$, $(\pm1,0)$. The origin is a center and $(\pm1,0)$ are saddles. Eq. (2.4) is a Hamiltonian system with the Hamiltonian function

$$H(u,v)=u^4-2u^2-2v^2. \tag{2.5}$$

Let $H(u,v)=k$, and when $k=-1$, it has a heteroclinic loop connected by the two critical points $(\pm 1,0)$, namely $v=\pm\sqrt{2}/2(u^2-1)$, so the corresponding kink wave and anti-kink wave solutions of the non-delay equation (1.3) exist. When $-1<k<0$, system (2.4) has a periodic orbit $v=\pm\sqrt{2}/2\sqrt{u^4-2u^2-k}$, $-\sqrt{1-\sqrt{1+k}}\le u\le\sqrt{1-\sqrt{1+k}}$, so the corresponding periodic wave solution of non-delay equation (1.3) exists.

To study the existence problem of the periodic orbit of the above ODE, we need the following Geometric Singular Perturbation Theorem [23-24].

Lemma 2.2 (Geometric Singular Perturbation Theorem). For the system

$$\begin{aligned} x'(t)&=f(x,y,\varepsilon),\\ y'(t)&=\varepsilon g(x,y,\varepsilon), \end{aligned} \tag{2.6}$$

where $x\in R^n, y\in R^l$ and ε is a real parameter, f,g are C^∞ on the set $V\times I$, where $V\in R^{n+l}$ and I is an open interval, containing 0. If when $\varepsilon=0$, the system has a compact, normally hyperbolic manifold of critical points M_0, which is contained in the set $\{f(x,y,0)=0\}$. Then for any $0<r<+\infty$, if $\varepsilon>0$, but sufficiently small, there exists a manifold M_ε:

(i) which is locally invariant under the flow of (2.6);

(ii) which is C^r in x,y and ε;

(iii) $M_\varepsilon=\{(x,y):x=h^\varepsilon(y)\}$ for some C^r function $h^\varepsilon(y)$ and y in some compact K;

(iv) there exist locally invariant stable and unstable manifolds $W^s(M_\varepsilon)$ and $W^u(M_\varepsilon)$ that lie within $O(\varepsilon)$, and are diffeomorphic to $W^s(M_0)$ and $W^u(M_0)$ respectively.

3. Existence of Solitary Wave of the Equation with Delay

Let $U(x,t)=\varphi(\xi)e^{i\theta}=\varphi(x-ct)e^{i(ax-wt)}$, and substituting it into Eq. (1.1), one gets

$$\begin{aligned} &w\varphi+\varphi''-a^2\varphi-(g*\varphi)\varphi^2-2\tau\varphi^2\varphi'=0,\\ &-c\varphi'+2a\varphi'=0, \end{aligned} \tag{3.1}$$

where

$$(g*\varphi)(\xi)=\int_0^\infty \frac{s}{\tau^2}e^{-\frac{s}{\tau}}\varphi(\xi+cs)ds. \tag{3.2}$$

Let $a=c/2$ and $\mu=w-c^2/4$, the system (3.1) with (3.2) is rewritten as

$$\varphi''-\mu\varphi-(g*\varphi)\varphi^2-2\tau\varphi^2\varphi'=0, \tag{3.3}$$

where $'$ denotes the derivative with respect to the variable ξ.

Taking $u=\varphi/\sqrt{\mu}$ and $z=\sqrt{\mu}\xi$ to Eq. (3.3) with (3.2), it becomes

$$\ddot{u}=-u+(g*u)u^2+2\tau\sqrt{\mu}u^2\dot{u}, \tag{3.4}$$

where $\cdot$ denotes the derivative with respect to the variable z and

$$(g*\varphi)(z)=\int_0^\infty \frac{s}{\tau^2}e^{-\frac{s}{\tau}}\varphi(\frac{z}{\sqrt{\mu}}+cs)ds. \tag{3.5}$$

Let $p(z)=(g*u)(z)$.

Differentiating p with respect to z, one gets that

$$\frac{dp}{dz}=\frac{1}{\sqrt{\mu}c\tau}(p-q), \tag{3.6}$$

where $q(z)=\int_0^{+\infty}\frac{1}{\tau}e^{-\frac{s}{\tau}}u(\frac{z}{\sqrt{\mu}}+cs)ds$.

Differentiating q with respect to z, one gets

$$\frac{dq}{dz}=\frac{1}{\sqrt{\mu}c\tau}(q-u). \tag{3.7}$$

Let $v=\dot{u}$, Eq. (3.4) can be rewritten as

$$\begin{aligned}
&\dot{u}=v,\\
&\dot{v}=-u+u^2p+2\tau\sqrt{\mu}u^2v,\\
&\sqrt{\mu}c\tau\dot{p}=p-q,\\
&\sqrt{\mu}c\tau\dot{q}=q-u.
\end{aligned} \tag{3.8}$$

When $\tau=0$, the system (3.8) becomes the following system

$$\begin{aligned}
&\dot{u}=v,\\
&\dot{v}=-u+u^2p,\\
&0=p-q,\\
&0=q-u.
\end{aligned}$$

Namely

$$\ddot{u}=-u+u^3. \tag{3.9}$$

When $\tau>0$, the system (3.8) determines a system of ODEs and its solutions exist in the four-dimensional (u,v,p,q) phase space in which system (3.9) has three critical points: $(0,0,0,0)$, $(1,0,1,1)$ and $(-1,0,-1,-1)$.

Let $z=\tau\eta$, the system (3.8) turns to the following fast system:

$$\begin{aligned}
&u'=\tau v,\\
&v'=\tau(-u+u^2p+2\tau\sqrt{\mu}u^2v),\\
&\sqrt{\mu}cp'=p-q,\\
&\sqrt{\mu}cq'=q-u.
\end{aligned} \tag{3.10}$$

where $'$denotes the derivative by η. If $\tau>0$, the slow system (3.8) and the fast system (3.10) are equivalent.

In the slow system (3.8), if $\tau=0$, the flow of this system is confined to the following set $M_0=\{(u,v,p,q)\in R^4, p=q=u\}$, which is a two-dimensional invariant manifold for the system (3.8). It's easy to obtain that M_0 is normally hyperbolic by the method of the linearization matrix [22, 26, 28]. According to the Geometric Singular Perturbation Theorem, there exists a locally invariant two-manifold M_τ with sufficiently small $\tau>0$, which can be expressed as

$$M_\tau=\{(u,v,p,q)\in R^4 : p=q+\phi(u,v,\tau), q=u+\psi(u,v,\tau)\}, \tag{3.11}$$

where ϕ,ψ depend smoothly on τ and satisfy $\phi(u,v,0)=\psi(u,v,0)=0$.

We expand the functions ϕ and ψ into the following form

$$\phi(u,v,\tau)=\tau\phi_1(u,v)+\tau^2\phi_2(u,v)+\cdots,$$
$$\psi(u,v,\tau)=\tau\psi_1(u,v)+\tau^2\psi_2(u,v)+\cdots. \quad (3.12)$$

Substituting (3.11) into the slow system (3.8), we get

$$\sqrt{\mu}c\tau\left[v+(\frac{\partial\phi}{\partial u}+\frac{\partial\psi}{\partial u})v+(\frac{\partial\phi}{\partial v}+\frac{\partial\psi}{\partial v})(-u+u^2(u+\phi+\psi)+2\tau\sqrt{\mu}u^2v)\right]=\phi,$$
$$\sqrt{\mu}c\tau\left[v+\frac{\partial\psi}{\partial u}v+\frac{\partial\psi}{\partial v}(-u+u^2(u+\phi+\psi)+2\tau\sqrt{\mu}u^2v)\right]=\psi. \quad (3.13)$$

Substituting (3.12) into (3.13), one gets $\phi_1=\sqrt{\mu}cv, \psi_1=\sqrt{\mu}cv.$

Thus (3.11) becomes

$$M_\tau=\{(u,v,p,q)\in R^4 : p=u+2\tau\sqrt{\mu}cv+O(\tau^2), q=u+\tau\sqrt{\mu}cv+O(\tau^2)\}. \quad (3.14)$$

Then the slow system (3.8) restricted to M_τ is written as

$$u'=v,$$
$$v'=-u+u^3+2\tau\sqrt{\mu}(c+1)u^2v+O(\tau^2). \quad (3.15)$$

When $\tau=0$, the system (3.15) reduces to the wave equation (2.4) of the corresponding non-delay system (1.3).

When $\tau=0$, the system (3.15) has a periodic loop L. Generally speaking, the periodic orbits will break as $\tau\neq 0$ and small. According to the Melnikov function method [26-28],

Let $d(\tau,L_1)=-\overrightarrow{n_1}\cdot\overrightarrow{M_\tau^sM_\tau^u}$, where $\overrightarrow{n_1}=(H_u(M_1),H_v(M_1))/\left|(H_v(M_1)-H_u(M_1))\right|$ [20-21], we obtain the following theorem.

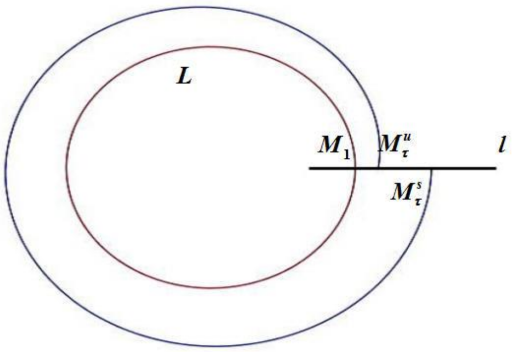

Figure 1. Orbits of system (3.15) for the case $0<\tau<<1$.

Theorem 3.1 For $\tau>0$, but sufficiently small, we have

$$d(\tau,L)=\tau\cdot N\cdot M(L)+O(\tau^2), \quad (3.16)$$

where $M(L)=\frac{8\sqrt{\mu}}{15}(c+1)h(k)+O(\tau)$, $h(k)>0, \forall k\in(-1,0)$ and $N>0$ is a constant.

Proof. Rewrite system (3.15) into

$$u'=v+\tau P(u,v),$$
$$v'=-u+u^3+\tau Q(u,v), \quad (3.17)$$

where

$$P(u,v)=0,$$
$$Q(u,v)=2\sqrt{\mu}(c+1)u^2v+O(\tau).$$

Form [20, 21, 27, 28] and noticing that system $(3.17)|_{\tau=0}$ is Hamiltonian, we have (3.16), and

$$M(L)=\int_L [H_v Q-(-H_u P)]X_i(s)ds=\int_L Q(u,v(u))du-\int_L P(u(v),v)dv,$$

where $X(s), -\infty < s < +\infty,$ is a parametric expression for L.

By lemma 2.1, we have the expression for L:

$$v^2(u)=\frac{1}{2}(u^4-2u^2-k), -\sqrt{1-\sqrt{1+k}} \le u \le \sqrt{1-\sqrt{1+k}},$$

Hence, $M(L)=\int_L Q(u,v(u))du=\frac{8\sqrt{\mu}}{15}(c+1)h(k)+O(\tau)$,

where $h(k)=-\frac{k}{\sqrt{1+\sqrt{1+k}}} EllipticK(\sqrt{\frac{1-\sqrt{1+k}}{1+\sqrt{1+k}}})+(3k+4)\sqrt{1+\sqrt{1+k}}(EllipticE(\sqrt{\frac{1-\sqrt{1+k}}{1+\sqrt{1+k}}}-EllipticK(\sqrt{\frac{1-\sqrt{1+k}}{1+\sqrt{1+k}}}))$

and $\mu=w-\frac{c^2}{4}$. By calculation of mathematical software, it can be proved that $h(k)>0$ when $-1<k<0$.

Furthermore, by noting that as $c=-1, \tau=0$, $M(L)=0$, we get

$$\frac{\partial M(L)}{\partial c}\Big|_{(c,\tau)=(-1,0)}=\frac{8}{15}\sqrt{\mu}, \mu=w-\frac{1}{4}\neq 0.$$

For $0<\tau<<1$, when $c\in U(-1)$, according to the Implicit Function Theorem, we have that there exists a function $c=-1+O(\tau)$ such that $d(\tau,L)=0$. From the definition of the function $d(\tau,L)$, we conclude that system (3.15) has a periodic loop orbit for $0<\tau<<1$. In other words, system (1.1) has a periodic loop wave solution for $0<\tau<<1$.

4. Conclusion

In this work, by the geometric singular perturbation theory and the Melnikov function method we establish existence of periodic loop wave solutions for the NLS equation with distributed delay having form (1.1) when $0<\tau<<1$, $c=-1+O(\tau)$.

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References

[1] N.N. Akhmediev, A.A. Ankiewicz, Solitons: *Nonlinear Pulses and Beams*,Chapman and Hall, London, 1997.

[2] C. Sulem, P. Sulem, *The Nonlinear Schrödinger Equation: Self-Focusing and Wave Collapse*, Springer, Berlin, 2000.

[3] G.P. Agrawal, *Nonlinear Fiber Optics*, Academic Press, San Diego, 1989.

[4] A. Hasegawa, Y. Kodama, *Solitons in Optical Communications*, Oxford University Press, New York, 1995.

[5] R.C. Davidson, *Methods in Nonlinear Plasma Theory*, Academic Press, New York, 1972.

[6] S. Qian, R. Zhu, *Nonlinear Optics*, Fudan University Press, Shanghai, 2005.

[7] M. Eslami, Solitary wave solutions for perturbed nonlinear Schrödinger's equation with Kerr law nonlinearity under the DAM, Optik, Vol. 126 (2015), p.1312–1317.

[8] H.Q. Zhang, New exact complex travelling wave solutions to nonlinear Schrödinger (NLS) equation, Commun. Nonlinear Sci. Numer. Simul., Vol. 14 (2009), p.668–673.
[9] D. Lu, A. Seadawy, M. Arshad, Applications of extended simple equation method on unstable nonlinear Schrödinger equations, Optik, Vol. 140 (2017), p.136–144.
[10] A. Seadawy, Exact solutions of a two-dimensional nonlinear Schrödinger equation, Appl. Math. Lett., Vol. 25 (2012), p. 687–691.
[11] Z. Zhang, Z. Liu, X. Miao, Y. Chen, Qualitative analysis and traveling wave solutions for the perturbed nonlinear Schrödinger's equation with Kerr law nonlinearity, Physics Letters A, Vol. 375 (2011), p. 1275–1280.
[12] R.M. El-Shiekh, A.G. Al-Nowehy, Integral methods to solve the variable coefficient NLSE, Z. Naturforsch, Vol. 68 (2013), p.225–260.
[13] M. Eslami, M. Mirzazadeh, B. Fathi Vajargah, Anjan Biswas, Optical solitons for the resonant nonlinear Schrödinger's equation with time-dependent coefficients by the first integral method. Optik, Vol. 125 (2014), p.3107–3116.
[14] E.M.E. Zayed , K.A.E. Alurrfi, On solving the nonlinear Schrödinger-Boussinesq equation and the hyperbolic Schrödinger equation by using the (G′/G , 1 /G) -expansion method, Int. J. Phys. Sci., Vol. 19 (2014), p.415–429.
[15] C.J. Chen, C.R. Menyuk, M.N. Islam, R.H. Stolen, Numerical study of the Raman effect and its impact on soliton-dragging logic gates, Opt. Lett., Vol. 16 (1991), No. 21, p. 1647–1649.
[16] C.F. Chen, B. Luo, The freeze of intrapulse Raman scattering effect of ultrashort solitons in optical fiber, Optik, Vol. 118 (2007), p.1–4.
[17] L. Yang, S.H. Liu, C.J. Liao, Steady solutions of optical soliton equation with nonlinear response delay term-static Kink solution, Acta Optica Sinica, Vol. 19(1999), No.6, p. 746–750.
[18] L. Yang, S.H. Liu, C.J. Liao, Steady solutions of optical soliton equation with nonlinear response delay term-drifting Kink solution, Acta Optica Sinica, Vol. 19(1999), No.6, p. 751–756.
[19] Z.H. Zhao, W.G. Ge, Traveling wave solutions for Schrödinger equation with distributed delay, Applied Mathematical Modeling, Vol. 35 (2011), p. 675–687.
[20] Y.H. Wu, M.A. Han, X.F. Chen, On the bifurcation of double homoclinic loops of a cubic system, Nonlinear Analysis, Vol. 68 (2008), p. 2487-2494.
[21] Y.H. Wu, Z.X. He, New model for secondary bifurcation to large limit cycles, Journal of Shanghai Jiaotong University，Vol. 10 (2005), p. 1727-1732.
[22] C.H. Xu, Y.H. Wu, L.X. Tian, B.L. Guo, On kink and anti-kink waves solution of Schrödinger equation with distributed delay, Journal of Applied Analysis and Computation, Vol.8(2018), No.5, p. 1385-1395.
[23] C.K.R.T. Jones, *Geometrical singluar perturbation theory*, in: R. Johnson (Ed.), Dynamical Systems, Lect. Notes Math.,Vol. 1609, Springer, New York, 1995.
[24] J. Li, K. Lu and P. W. Bates, Geometric singular perturbation theory with real noise, J. Diff. Eqs., Vol. 259 (2015), p. 5137-5167.
[25] N. Fenichel, Geometric singluar perturbation theory for ordinary differential equations, J. Diff. Eqs., Vol. 31(1979), p. 53-98.
[26] J. Li, *Chaos and Melnikov method*, Chongqin University Press, 1989.
[27] M.A. Han, *Bifurcation Theory and Periodical Solution of Dynamic System*, Science Press, Beijing, 2002.
[28] M.A. Han, D. Zhu, *Bifurcation Theory of Different Equation*, Coal Mine Industry Publishing House, Beijing, 1994.