Symmetry Transformations and Explicit Solutions of a (1+1)-dimensional KdV-type Equation with Variable Coefficients

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Abstract

With the aid of Maple, we use Lou's direct method to study a (1+1)-dimensional KdV-type equation with variable coefficients. And we give the symmetry transformations and exact solutions of the KdV-type equation.

Keywords

Symmetry Transformation; Differential Equations with Variable Coefficient; Lou's Direct Method; Exact Solution.

1. Introduction

Differential equations with variable coefficients have more physical backgrounds and practical significances because of its coefficients' arbitrariness. Recently, scientists have become more interested in symmetric properties, construction of exact solutions and corresponding physical phenomenon of differential equations with variable coefficients. Many effective methods have been applied to such equations successively [1-3].

In this paper we investigate a (1+1)-dimensional KdV-type equation with variable coefficients by Lou's direct method [4-5]. The plan of the present paper is as follows: Section 2 presents two sets of symmetric transformations and exact solutions of the KdV-type equation, and gives the corresponding numerical examples. Section 3 gives a short summary.

2. Symmetry reductions and exact solutions of the KdV-type equation

This paper will consider the following KdV-type equation with variable coefficients,

$$u_t + uu_x + a(t)u_x + b(t)u_{xxx} = 0, \tag{1}$$

where u is a function about x, t, and the coefficients $a(t), b(t)$ are differentiable functions about t. Replacing the functions $a(t), b(t)$ with constants k_1, k_2, we have

$$u_t + uu_x + k_1 u_x + k_2 u_{xxx} = 0. \tag{2}$$

In order to obtain the symmetric transformations of Eq. (1), we assume

$$u = A + BU(X, T), \tag{3}$$

where A, B, U, X, T are functions of x, t, and U satisfies the same form as Eq. (2) but with the new independent variables,

$$U_T + UU_X + k_1 U_X + k_2 U_{XXX} = 0. \tag{4}$$

Substituting (3) into Eq. (1), and eliminating U_{XXX} by Eq. (4), we have

$$b(t)BT_x^{\,3}U_{TTT} + V(x, t, U) = 0, \tag{5}$$

where V is a complex function which is independent of U_{TTT}. Eq.(5) holds for arbitrary solution U, if and only if all coefficients of the derivatives of U are zero. Obviously, $\mathrm{b}(t)BT_x^3 = 0$, without loss of generality we assume that

$$T = T(t). \tag{6}$$

Using Eq.(6) to reduce Eq.(5), collect the coefficients of the derivatives of U and constant terms, we can get the determining equations about A, B, X, T,

$$\begin{aligned}
&A_t + AA_x + a(t)A_x + b(t)A_{xxx} = 0,\\
&B_t + AB_x + A_xB + a(t)B_x + b(t)B_{xxx} = 0,\\
&BX_t + ABX_x + a(t)BX_x + 3b(t)B_{xx}X_x + 3b(t)B_xX_{xx} - \frac{b(t)Bk_1X_x^3}{k_2} + b(t)BX_{xxx} = 0,\\
&BB_x = 0, \qquad 3b(t)(B_xX_x^2 + BX_xX_{xx}) = 0,\\
&B^2X_x - b(t)\frac{BX_x^3}{k_2} = 0, \qquad BT_t - b(t)\frac{BX_x^3}{k_2} = 0.
\end{aligned} \tag{7}$$

With the aid of symbolic software Maple, we can get two cases as follows.

Case 1:

$$\begin{aligned}
&A(x,t) = d_4, B(t) = d_1, T(t) = d_1d_2t + d_3, \alpha(t) = \frac{-(\frac{d}{dt}f(t)) + (d_1k_1 - d_4)d_2}{d_2},\\
&X(x,t) = d_2x + f(t), b(t) = \frac{d_1k_2}{d_2^2},
\end{aligned} \tag{8}$$

where $d_i(i = 1,2,3,4)$ are arbitrary constants, and $f(t)$ is an arbitrary function. The KdV-type equation (1) can be reduced to

$$u_t + uu_x + \frac{-(\frac{d}{dt}f(t)) + (d_1k_1 - d_4)d_2}{d_2}u_x + \frac{d_1k_2}{d_2^2}u_{xxx} = 0. \tag{9}$$

The symmetry transformation of Eq.(9) is

$$u = d_4 + d_1U\text{（}d_2x + f(t)\text{，} d_1d_2t + d_3\text{）}. \tag{10}$$

It is easy to get a exact solution of Eq. (4) ,

$$U(X,T) = -12k_2C_2^2\tanh(C_2x + C_3t + C_1)^2 - \frac{-8C_2^3k_2 + C_2k_1 + C_3}{C_2}, \tag{11}$$

where $C_i(i = 1,2,3)$ are arbitrary constants.

We can get an exact solution of Eq. (9) by combining (10) and (11),

$$u(x,t) = d_4 + d_1\left\{-12k_2C_2^2\tanh\left[C_2(d_2x + f(t)) + C_3(d_1d_2t + d_3) + d_1\right]^2 - \frac{-8C_2^3k_2 + C_2k_1 + C_3}{C_2}\right\}. \tag{12}$$

Next, we consider the evolution of the exact solution (12).

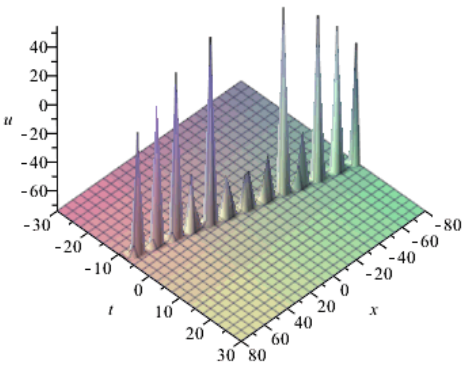

Fig. 1 $C_1 = 3, C_2 = 1, C_3 = 2,$ $d_1 = 5, d_2 = 1, d_3 = 1, d_4 = 1,$ $k_1 = 1, k_2 = 3, k_3 = 2,\ f(t) = \sin(t).$

A solution (12) of u .

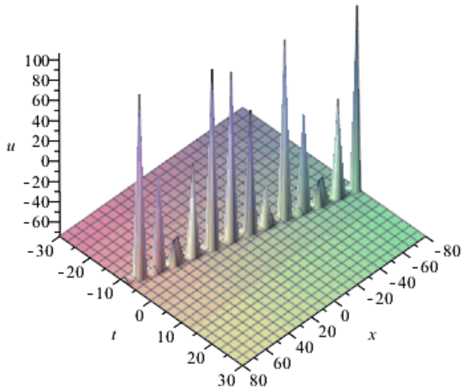

Fig. 2 $C_1 = 3, C_2 = 1, C_3 = 2,$ $d_1 = 5, d_2 = 1, d_3 = 1, d_4 = 1,$ $k_1 = 1, k_2 = 3, k_3 = 2,\ f(t) = \mathrm{sech}(t).$

A solution (12) of u .

Case 2:

$$A(x,t) = \frac{d_1 x}{d_1 t + d_2} + \frac{d_1 f(t)}{d_3} - \frac{k_1}{d_1 t + d_2} + d_5, \alpha(t) = -\frac{(d_1 t + d_2)(\frac{d}{dt} f(t)) + d_1 f(t) + d_5 d_3}{d_3}, \tag{13}$$
$$B(t) = -\frac{1}{d_1 t + d_2}, b(t) = -\frac{(d_1 t + d_2)k_2}{d_3^2}, T(t) = \frac{d_3}{(d_1 t + d_2)d_1} + d_4, X(x,t) = \frac{d_3 x}{d_1 t + d_2} + f(t).$$

Eq. (1) can be reduced to

$$u_t + uu_x + -\frac{(d_1 t + d_2)(\frac{d}{dt} f(t)) + d_1 f(t) + d_5 d_3}{d_3} u_x - \frac{(d_1 t + d_2)k_2}{d_3^2} u_{xxx} = 0, \tag{14}$$

and the symmetry transformation of Eq.(14) is

$$u = \frac{d_1 x}{d_1 t + d_2} + \frac{d_1 f(t)}{d_3} - \frac{k_1}{d_1 t + d_2} + d_5 - \frac{1}{d_1 t + d_2} U(\frac{d_3 x}{d_1 t + d_2} + f(t), \frac{d_3}{(d_1 t + d_2)d_1} + d_4). \tag{15}$$

We obtain an exact solution of Eq. (14) by combining (11) and (15),

$$u(x,t) = \frac{d_1 x}{d_1 t + d_2} + \frac{d_1 f(t)}{d_3} - \frac{k_1}{d_1 t + d_2} + d_5 - \frac{-12k_2 C_2^2 \tanh\left(C_2\left(\frac{d_3 x}{d_1 t + d_2} + f(t)\right) + C_3(\frac{d_3}{(d_1 t + d_2)d_1} + d_4) + C_1\right)^2 - \frac{-8C_2^3 k_2 + C_2 k_1 + C_3}{C_2}}{d_1 t + d_2}. \tag{16}$$

Next, we consider the evolution of the exact solution (16).

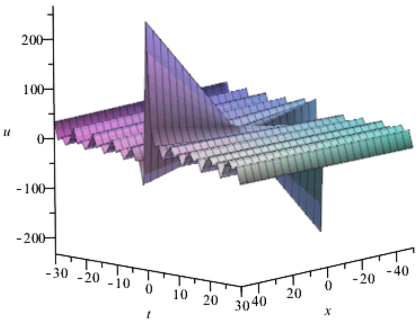

Fig. 3 $C_1=3, C_2=1, C_3=2,$ $d_1=5, d_2=1, d_3=1, d_4=1, d_5=3$ $k_1=1, k_2=3, k_3=2$, $f(t)=\tan(t)$.

A solution (16) of u.

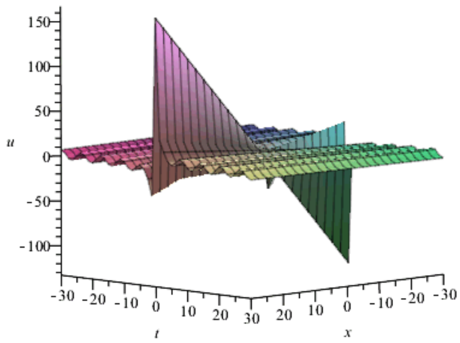

Fig. 4 $C_1=3, C_2=1, C_3=2,$ $d_1=5, d_2=1, d_3=1, d_4=1, d_5=3,$ $k_1=1, k_2=3, k_3=2$, $f(t)=\sin(t)$.

A solution (16) of u.

3. Summary

In this paper we study a (1+1)-dimensional KdV-type equation with variable coefficients. We obtain the symmetry transformations and exact solutions of the KdV-type equation by Lou's direct method. Moreover, we establish the transformation between the solutions of differential equations with constant coefficients and the ones with variable coefficients.

Acknowledgments

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