Equilibria Analysis of One - dimensional Hopfield Neural Networks

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Abstract

In this paper, we introduce the one-dimensional Hopfield neural network firstly and then analyze the existence for the equilibria of one-dimensional Hopfield neural network. By using the geometric method, we conclude the sufficient conditions of the networks having one, two or three equilibria.

Keywords

Hopfield neural network, equilibria, geometric method.

1. Introduction

Hopfield neural network model was proposed by the American physicist J.J.Hopfield in 1984, the model and its promotion have attracted much attention in the fields of optimization, pattern recognition, associative memory and so on[1-4]. Neural network model has multiple equilibria which is the necessary condition to apply the model to associative memory and pattern recognition, the related work has also been a lot of attention[5-6]. In this paper, we study the following one-dimensional Hopfield neural network, using the nonlinear differential equation as follows:

$$\frac{du}{dt} = -Cu + Tg(u) + I \tag{1}$$

Where $u(\mathrm{t})$ represents the state variables of neurons, C indicates the rate at which neurons recover in isolated resting state, T represents the neuron's own right of communication, I indicates external input, $g(u)$ is a continuous function on the real number field R, called the activation function of neurons.

In Hopfield neural network, the activation function is continuous and non-linear sigmoid function generally, which satisfies the following conditions:

$$\begin{cases} \lim\limits_{u\to\pm\infty} g(u) = \pm 1 \\ 0 < \dot{g}(u) < \dot{g}(0), \quad u \in \mathrm{R} - \{0\} \\ \left|\dot{g}(u)\right| \le 1, \quad u \in \mathrm{R} \\ \lim\limits_{u\to\pm\infty} \dot{g}(u) = 0 \end{cases} \tag{2}$$

The sufficient conditions under the cases of one equilibrum, two equilibria and three equilibria are analyzed by using the geometric method.

2. Main results

The situation $CT = 0$ has been well studied, the following C, T are non-zero real numbers.

Theorem 1 If the activation function $g(\mathrm{u})$ is a sigmoid function satisfying the formula (2), then there is at least one equilibrum in the system (1).

Proof: First, let's suppose $C > 0$, we have

$$\lim_{\mathrm{u}\to+\infty} Cu - I = +\infty, \lim_{\mathrm{u}\to-\infty} Cu - I = -\infty.$$

Since $g(u)$ is a continuous function, and $\lim_{u\to\pm\infty} g(u) = \pm 1$, then $Tg(u)$ is bounded. Let $F(u) = Tg(u) - (Cu - I)$, we get

$$\lim_{u\to+\infty} F(u) = -\infty,$$
$$\lim_{u\to-\infty} F(u) = +\infty.$$

Namely, there exist $M_1 > 0$, when $u > M_1$, we have $F(u) < 0$. And there exist $M_2 < 0$, when $u < M_2$, we have $F(u) > 0$. So the function $F(u)$ has zero by it's continuity. Clearly, when $C < 0$, $F(u)$ also has zero. In summary, there exists u^* such that $-Cu^* + Tg(u^*) + I = 0$, that is, the equilibria of the system (1) are exist. The proof is completed.

Note 1: For the sake of demonstration, the activation functions $g(\text{u})$ of the systems (1) satisfy the formula (2).

Theorem 2 Let $f_1(u) = C - Tg'(u), u > 0$ and $f_2(u) = C - Tg'(u), u < 0$, then the inverse functions of $f_1(u)$ and $f_2(u)$ are exist respectively.

Proof: According to the properties of the sigmoid function, when $u > 0$, $g'(u)$ is monotonous function, when $u < 0$, $g'(u)$ is also monotonous function, then according to the existence theorem of inverse functions, the inverse functions of $f_1(u)$ and $f_2(u)$ are exist.

Note 2: To prove convenience, we denote the inverse functions of $f_1(u)$ and $f_2(u)$ as $f_1^{-1}(u)$ and $f_2^{-1}(u)$ respectively.

Theorem 3 Let

$$I_1 = \max\left\{Cf_1^{-1}(0) - Tg(f_1^{-1}(0)), Cf_2^{-1}(0) - Tg(f_2^{-1}(0))\right\},$$
$$I_2 = \min\left\{Cf_1^{-1}(0) - Tg(f_1^{-1}(0)), Cf_2^{-1}(0) - Tg(f_2^{-1}(0))\right\}.$$

Then we have the following conclusion:

(1) When $CT < 0$, the system (1) has the only one equilibrium;

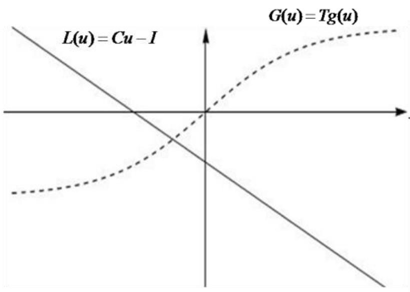
(2) When $CT > 0$, there are three kinds of situations:

① If $I < I_2$ *or* $I > I_1$, the system (1) has the only one equilibrium;

② If $I = I_2$ *or* $I = I_1$, the system (1) have two equilibria;

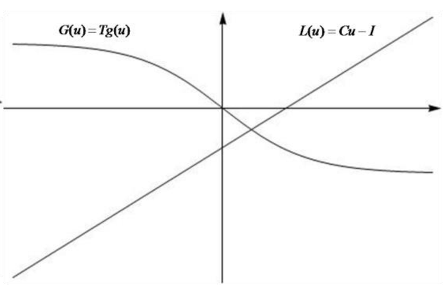
③ If $I_2 < I < I_1$, the system (1) have three equilibria;

Proof: To find the equilibria of the system (1), Let $-Cu + Tg(u) + I = 0$, that is $Tg(u) = Cu - I$. Let $G(u) = Tg(u)$, $L(u) = Cu - I$. That is to say, the equilibria of the system (1) is actually the intersection of the curve $G(u)$ and the line $L(u)$. There are four different situations, see Fig. 1-Fig. 4.



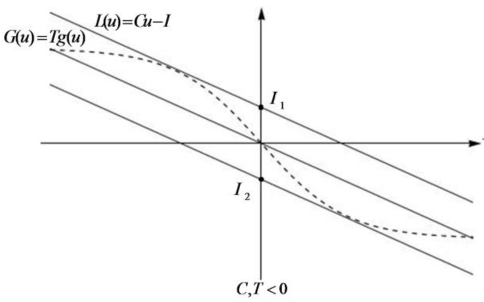
$C < 0, T > 0$

Fig.1



$C > 0, T < 0$

Fig.2



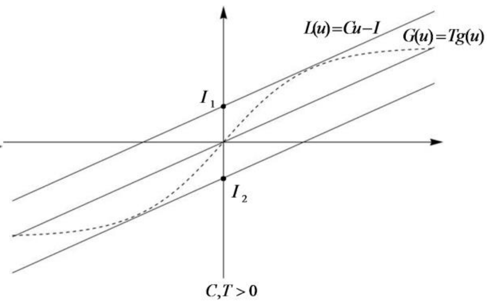


Fig.3 Fig.4

(1) When $CT<0$, $G(u)$ and $L(u)$ have the opposite monotonicity, as shown in Fig.1 or Fig.2. So $F(u)$ is monotonous. According to Theorem 1, $F(u)$ has the only zero by the monotonicity. In summary, when $CT<0$, the system (1) has the only equilibrium.

(2) When $CT>0$, $G(u)$ and $L(u)$ have the same monotonicity, as shown in Fig.3 or Fig.4. To find the intersection of the curve $G(u)$ and the line $L(u)$, we consider the critical situation the curve $G(u)$ is tangent to the line $L(u)$ firstly.

If the curve $G(u)$ is tangent to the line $L(u)$, there are two intersections between the curve $G(u)$ and the straight line $L(u)$ obviously. One of the intersections is their tangent point, and the tangent point satisfies the equations:

$$\begin{cases} Tg(u)=Cu-I \\ Tg'(u)=C \end{cases}$$

We can get $I=I_1$ or I_2 from the above equations, where

$$I_1=\max\left\{Cf_1^{-1}(0)-Tg(f_1^{-1}(0)),Cf_2^{-1}(0)-Tg(f_2^{-1}(0))\right\},$$
$$I_2=\min\left\{Cf_1^{-1}(0)-Tg(f_1^{-1}(0)),Cf_2^{-1}(0)-Tg(f_2^{-1}(0))\right\}.$$

So, when $I=I_2$ *or* $I=I_1$, the curve $G(u)$ and the line $L(u)$ have two intersections, the system (1) has two equilibria. From the features of the figures of the curve $G(u)$ and the line $L(u)$, we can get the following conclusion: when $I<I_2$ *or* $I>I_1$, the curve $G(u)$ and the line $L(u)$ have only one intersection, so the system (1) has only one equilibrum. When $I_2<I<I_1$, the curve $G(u)$ and the line $L(u)$ have three intersections, the system (1) has three equilibria.

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