A Poisson type capacity on a class of Poisson manifolds

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Abstract

This paper introduces a Poisson analogy of symplectic capacities on a class of Poisson manifolds with the help of studying the geometry structure of the Poisson manifolds. The embedding and conformal properties are proved, the finiteness of Poisson capacity is also given on standard Poisson space.

Keywords

Poisson capacity, poisson system, Hamiltonian diffeomorphism.

1. Introduction

This paper is devoted to establishing a capacity on Poisson manifolds. In symplectic geometry, symplectic capacity plays an important role in studying symplectic embedding problem, searching for periodic solutions on convex energy surfaces. There are some celebrated capacities in symplectic case, for example, I. Ekeland and H. Hofer define a class of symplectic invariants for subsets of $\mathbb{R}^{2n}$ [2,3], Gromov gives the concept of symplectic width[8], H. Hofer and E. Zehnder extend symplectic invariants to general symplectic manifold [4], M. Jiang computes the Hofer-Zehnder symplectic capaticy for two dimensional manifolds[9], C. Liu and Q. Wang study the symmetrical symplectic capacity and give its applications on symmetrical Hamiltonian system[7]. Poisson manifold is a manifold with Poisson structure. A manifold M is called Poisson manifold, if there exists a Poisson bracket $\{\ \}$ on the sets of smooth functions on the manifold. Any smooth function f, g, h on the manifold satisfies the following:

$$\{f,g\}=-\{g,f\}$$

$$\{f,gh\}=g\{f,h\}+h\{f,g\}$$

$$\{f,\{g,h\}\}+\{h,\{f,g\}\}+\{g,\{h,f\}\}=0 \tag{1}$$

Poisson, Jacobi, Lie, Marsden, Weinstein and other mathematicians have contributed a lot in this field [1]. There are many similar notations in the researching of Poisson geometry, such as Poisson diffeomorphism, Poisson vector, Poisson Hamiltonian flow, but there are few results about Poisson capacity on Poisson manifold. In this paper we will give a Poisson analogy of capacities with the help of studying the geometry structure of the Poisson manifolds. The main results of this paper are the following:

Theorem 1. If M are regular Poisson manifolds of type S, then there function $c(M)=\sup\limits_{l_\alpha} c_s(l_\alpha)$ satisfies the properties of capacity.

Theorem 2. On standard Poisson space $(\mathbb{R}^n,\{\ \}_0)$, the function $c((\mathbb{R}^n,\{\ \}_0))=\sup\limits_{l_\alpha} c_s(l_\alpha)$ is finite.

2. Preliminaries

In this section, we will give some basic definitions and properties of symplectic and Poisson geometry, details can be found in [1,5,6,10].

Definition 3. Let $\left(M,\{\ \}\right)$ be a Poisson manifold, a diffeomorphism ϕ of M is called Poisson if it keeps the Poisson bracket, that is to say, for any smooth functions f,g on M, it satisfies

$$\phi^*\{f,g\}=\{f\circ\phi,g\circ\phi\} \tag{2}$$

If the manifold is symplectic, a diffeomorphism ϕ is called symplectic if it keeps the symplectic form.

Definition 4[5]. A symplectic capacity is a map which associates with every symplectic manifold (M,ω) a nonnegative number or infinity satisfying the following properties:

$$c(M,\omega)\le c(N,\tau) \tag{3}$$

If there exists a symplectic embedding $\varphi:(M,\omega)\to(N,\tau)$. For all real number $\alpha\ne 0$,

$$c(M,\alpha\omega)=\left|\alpha\right|c(N,\tau) \tag{4}$$

For the open ball unit ball $B(1)$ and the symplectic cylinder $Z(1)$ in the standard space $(\mathbb{R}^{2n},\omega_0)$,

$$c(B(1),\omega_0)=c(Z(1),\omega_0)=\pi \tag{5}$$

The ball and the cylinder in $(\mathbb{R}^{2n},\omega_0)$ is defineded as following:

$$B(r)=\{(x,y)|\ \left|x^2+y^2<r^2\right|\}$$

$$Z(r)=\{(x,y)|\ \left|{x_1}^2+{y_1}^2<r^2\right|\} \tag{6}$$

Here $(x,y)\in\mathbb{R}^{2n}$ are the symplectic coordinates.

Since there is no variational structure in general Poisson dynamical system, it is difficult to define capacity by the usual symplectic method. Note that Poisson manifold is a disjoint union of symplectic leaves, we may give the definition with the help of the symplectic leaves.

Definition 5. We call a Poisson manifold is type S if it satisfies the following: the Poisson manifold is closed or open, and all the symplectic leaves are closed or open. If the Poisson manifold is open, we assume that all the Hamiltonian functions are compactly supported and all the symplectic leaves are open or closed, and the Hamiltonian functions restricted to the leaves are still compactly supported when the symplectic leaves are open.

By the assumption of the Poisson manifold, we know that the solution of the following equation always exists

$$\dot{x}=X_{H_t}(t,x)$$

$$X_{H_t}=\left\{\bullet,H_t\right\} \tag{7}$$

If the flow of the Hamiltonian vector globally exists, we call the vector complete, and call the Hamiltonian function complete function. If all the Hamiltonian functions are complete, we call the Poisson manifold complete.

Definition 6. A Poisson manifold is regular, if all the rank of the symplectic leaves are the same.

Definition 7[10]. Let $(M,\{\})$ be a Poisson manifold, and Π be a Poisson bi-vector, we difine the sharp map by:

$$\#\Pi:T^*M\to TM$$

$$\#_{\Pi,x}(x,\alpha_x)=(x,i_{\alpha_x}\Pi(x)),\quad \alpha_x\in {T_x}^*M \tag{8}$$

Definition 8[10]. The rank of te Poisson manifold at the point x is defineded as following $\rho\left(x\right)=\dim im\#_{\Pi,x}$

Definition 9. A Poisson capacity is a map on the sets of Poisson manifold, it is a nonnegative number or infinity satisfying the following properties:

$$c(M,\{\}) \le c(N,\{\}) \tag{9}$$

If there exists a Poisson embedding $\varphi:(M,\{\}) \to (N,\{\})$. For all real number $\alpha \ne 0$,

$$c(M,\alpha\{\}) = |\alpha| c(M,\{\}) \tag{10}$$

For the open unit ball $B(1)$ and the Poisson cylinder $Z(1)$ in the standard space $\mathbb{R}^n(n \ge 2)$, $c(B(1),\{\ \}_0)$ and $c(Z(1),\{\ \}_0)$ are finite. The ball and the cylinder in $\mathbb{R}^n(n \ge 2)$ is defineded as following:

$$B(r) = \{(x,y) |\ |x^2 + y^2| < r^2\}$$

$$Z(r) = \{(x,y) |\ |x_1^2 + y_1^2| < r^2\} \tag{11}$$

Here $(x,y) \in \mathbb{R}^n$ are the basic coordinates.

We should make more explanations on the standard Poisson space $(\mathbb{R}^n,\{\ \}_0)$, if the dimension of the Poisson manifold is $2m$, then the standard Poisson manifold is just the standard symplectic space $(\mathbb{R}^{2n},\omega_0)$. If the dimension of the Poisson manifold is $2m$+1, then the Poisson manifold can be viewed as the product of the $2m$ dimensional symplectic space $\mathbb{R}^{2n}$ and the 1dimensional space $\mathbb{R}$. For every point in the the 1dimensional space $\mathbb{R}$, the symplectic leaf is $2m$ dimensional subspace contained the point, the restriction of the Poisson bracket on symplectic space is the usual bracket generated by the standard symplectic form ω_0, the restriction of the Poisson bracket on the left space $\mathbb{R}$ is 0.

3. Proof of main results

In this part we will give the construction of the Poisson capacity. Suppose the symplectic leaves of the Poisson manifold are all closed or open, and the Poisson manifold is regular, that is to say, there is no singularities on the Poisson manifold. With the help of the geometry structure of the Poisson manifold, we now can give the construction of the Poisson capacity.

Denote by $l_\alpha \in M$ the symplectic leaves of the Poisson manifold, suppose there is a capacity c_s on the symplectic manifold, then we define

$$c(M) = \sup_{l_\alpha} c_s(l_\alpha) \tag{12}$$

Theorem 1. If M are regular Poisson manifolds of type S, then there function $c(M) = \sup\limits_{l_\alpha} c_s(l_\alpha)$ satisfies the properties of capacity. In order to prove Theorem 1, we need the following lemma.

Lemma 2[6]. Let $f: P_1 \to P_2$ be a Poisson map and H be a Hamiltonian function on P_2. If φ_t is the flow of X_H and ψ_t is the flow of $X_{H\circ f}$, then we have the following

$$\varphi_t \circ f = f \circ \psi_t$$

$$T_f \circ X_{H\circ f} = X_H \circ f \tag{13}$$

Proof of Theorem 1: We need prove the function c satisfies the properties of the Poisson capacity.

Assume that there exists a Poisson embedding $\varphi:(M,\{\}) \to (N,\{\})$, if f is a Poisson diffeomorphism of manifold N, then $\varphi^* f$ is a Poisson diffeomorphism of manifold M. Since the symplectic leaf is the sets of points which can be connected by Hamiltonian flow, we know that if f_t is a Hamiltonian flow on N, then $\varphi^* f_t$ is a Hamiltonian flow on M, the diffeomorphism maps leaf of M to leaf of N. For symplectic leaves $l_\alpha M$ and $l_\alpha N$ of Poisson manifolds M and N, by the assumption of the symplectic leaves and the properties of Hamiltonian functions on the leaves, we know that all the symplectic leaves are closed or open, if the Poisson manifold is open, all the Hamiltonian functions are compactly supported, and the Hamiltonian functions restricted to the leaves are still compactly supported when the leaves are open. So we can use the symplectic capacities on the leaves of the poisson manfold, symplectic capacity satisfies the following properties:

$$c(l_\alpha M,\omega_\alpha)\le c(l_\alpha N,\tau_\alpha) \tag{14}$$

Here ω_α τ_α is the corresponding symplectic forms on symplectic leaves $l_\alpha M$ and $l_\alpha N$. By the definition of Poisson capacity,

$$c(M)=\sup_{l_\alpha} c_s(l_\alpha M)$$
$$c(N)=\sup_{l_\alpha} c_s(l_\alpha N) \tag{15}$$

So we have the embedding properties

$$c(M)\le c(N) \tag{16}$$

For the conformal properties, we also use the symplectic leaves structure to prove it. For symplectic leaves $l_\alpha M$ of Poisson manifold M , by the conformal properties of the symplectic capacity, we have

$$c(l_\alpha M,\lambda\omega_\alpha)=|\lambda| c(l_\alpha M,\omega_\alpha) \tag{17}$$

The above equality holds on every symplectic leaf, following the definition of Poisson capacity, we can get

$$c(M,\alpha\{\})=|\alpha| c(M,\{\}) \tag{18}$$

Proof of Theorem 2:

For the finiteness of the capacity of the open unit ball $B(1)$ and the Poisson cylinder $Z(1)$ in the standard space $\mathbb{R}^n(n\ge 2)$, we will use the assumptions on the standard Poisson space. If the dimension of the Poisson manifold is $2m$, then the standard Poisson manifold is the standard symplectic space $(\mathbb{R}^{2n},\omega_0)$, the symplectic open unit ball $B(1)$ and cylinder $Z(1)$ is finite. If the dimension of the Poisson manifold is $2m+1$, according to the assumptions of the Poisson structure, we can assume that the coordinates of the Poisson space is $(x_1,x_2,\cdots x_m,y_1,y_2\cdots y_m,z_{2m+1})$, the symplectic leaves are the $2m$ dimensional subspace orthogonal with the last one dimensional space.

For the fixed $z_{2m+1}(0)$, we compute the capacity of the the open unit ball $B(1)$ and the Poisson cylinder $Z(1)$.

Case 1: for the open unit ball $B(1)$, when the last coordinate is fixed, the symplectic leaves of the ball are the $2m$ dimensional balls

$$B_{2\mathrm{m}}(t)=\{(x_1,\cdots x_m,y_1,\cdots y_m)|\ |x^2+y^2|<\mathrm{t}^2\} \tag{19}$$

By the finiteness of the symplectic unit ball, each ball has finte symplectic capacity, and the radius is less than 1, so the supremum of the capacities are finite, we can get the finiteness of the open Poisson unit ball.

Case 2: for the Poisson cylinder $Z(1)$, when the last coordinate is fixed, the symplectic leaves of the Poisson cylinder are the usual symplectic cylinders

$$Z(t)=\{(x_1,\cdots x_m,y_1,\cdots y_m,\mathrm{t})|\ |x_1{}^2+y_1{}^2|<1\} \tag{20}$$

Use the property of the symplectic cylinder $Z(1)$ again, the supremum of the capacities are also finite, we can get the finiteness of the Poisson cylinder $Z(1)$.

4. Conclusion

In this paper we introduce a Poisson type capacity on a class of Poisson manifold by the geometry of the structure. Using the finiteness of symplectic properties on the standard symplectic space, we prove the finiteness of the Poisson capacity on standard Poisson space. The capacity on Poisson manifold should have some application in the studying of Poisson dynamics, we will study the other capacities and its applications in the future.

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