The lower bound of number of points for fundamental systems in spherical t-designs

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Abstract

In this paper, we study the fundamental systems of in spherical t-design when N = 2t + 1. Therefore, we present a lemma to connect the fundamental system with spherical t-design.

Keywords

Spherical t-design; Fundamental system.

1. Introduction

Considering 2-dimensional unit sphere $\mathbb{S}^2$. The concept of spherical t-design was introduced by Delsarte et al. [1].

Definition 1.1 A finite set $X_N := \{x_1, \ldots, x_N\}$ is a spherical t-design, then will satisfy

$$\frac{1}{N}\sum_{i=1}^{N} p(x_i) = \frac{1}{4\pi}\int_{\mathbb{S}^2} p(x)\mathrm{d}\omega(x) \qquad \forall p \in \prod_t, \tag{1}$$

where $\prod_t := \prod_t(\mathbb{S}^2)$ is the space of spherical polynomials on $\mathbb{S}^2$ with degree at most t and $\mathrm{d}\omega(x)$ denotes the surface measure on $\mathbb{S}^2$.

A lower bound on the number of points N to construct a spherical t-design for any $t \geq 1$ on $\mathbb{S}^2$ was given in [1]. From then on, the relation between t and N has been studied extensively [2-6].

Finding spherical t-design has been expressed as equivalent conditions [7,8]. Fundamental spherical t-design has been studied in [8] for certain non-linear equation systems. However, the lower bound of N to construct a fundamental spherical t-design remains to observe. In this paper, we find the lower bound of N for fundamental system in spherical t-design based on the work [9].

The paper is organized as follows. In the next section, we introduce the required knowledge. In section 3, the non-existence of fundamental system of degree t in spherical t-design is proved and the conditions of fundamental spherical t-design is given. Section 4 ends this paper with a brief conclusion.

2. Preliminaries

The addition theorem for spherical harmonics on $\mathbb{S}^2$ gives

$$\sum_{k=1}^{2n+1} \mathrm{Y}_n^k(x)\mathrm{Y}_n^k(y) = \frac{2n+1}{4\pi}\mathrm{P}_n(<x,y>) \quad \forall x, y \in \mathbb{S}^2. \tag{2}$$

Sloan and Womersley [7] studied (2) and presented an important proposition of spherical t-design.

Proposition 2.1 A finite set $X_N := \{x_1, \ldots, x_N\}$ is a spherical t-design if and only if

$$\sum_{i=1}^{N} \mathrm{Y}_n^k(x_i)=0, \quad k=1,\ldots,2n+1 \;\; n=1,\ldots,t. \tag{3}$$

And then, Dai and Xu [9] gave some properties to these spaces and founded the fundamental system of a certain degree on the sphere. Let $\{x_1,\ldots,x_N\}$ be a finite set of point on $\mathbb{S}^2$, where $N \geq 2n+1$. We define matrices M_k similar to [9], presented as

$$M_k = \begin{bmatrix} Y_n^1(x_1) & \ldots & Y_n^1(x_k) \\ \vdots & \ddots & \vdots \\ Y_n^k(x_1) & \ldots & Y_n^k(x_k) \end{bmatrix}, \tag{4}$$

For k=1,...2n+1.

3. Main Results

According to the theorem and lemma from [9], we obtain the lemma 3.1 immediately.

Lemma 3.1 A finite set $\mathrm{X}_N := \{x_1,\ldots,x_N\}$ is called a fundamental system of degree n for $\mathbb{H}_n$ on the sphere $\mathbb{S}^2$ if

$$\prod_{i=0}^{n} \det(M_{2i+1}^T M_{2i+1}) > 0.$$

Proof. Firstly, $\{Y_i^1,\ldots,Y_i^{2i+1}\}$ is a set of basis for $\mathbb{H}_i$ for i=0,…,n. Secondly, we know that $\prod_n = H_0 \oplus H_1 \oplus \cdots H_n$. Thus, there exist a set of basis $\{Y_0^1,\ldots,Y_n^{2n+1}\}$ to generate a fundamental system X_N with $N \geq 2n+1$ for $\prod_n$. Suppose X_N is a fundamental system for $\prod_n$. According to [9], we know $\det(M_{2i+1}) \neq 0$ for i=0,…,n. Therefore $\prod_{i=0}^{n} \det(M_{2i+1}^T M_{2i+1}) > 0.$ and the proof is complete.

By studying the fundamental system, we give a proposition to think of the lower bound of fundamental system in spherical t-design.

Proposition 3.2 Suppose a finite set $\mathrm{X}_N := \{x_1,\ldots,x_N\}$ with N=2t+1 and t>1 is a spherical t-design. Then X_N is not a fundamental system of degree t.

Proof. According to Proposition 2.1, if X_N is a spherical t-design, then by the property of determinant we have $\det(M_{2t+1}) = 0$. Hence we complete the proof.

4. Conclusion

In this paper, we discover the fundamental system in spherical t-design, and presented a lemma to connect them.

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