A generalized regularization method for solving linear ill-posed problems under general source conditions

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Abstract

In this paper we consider linear ill-posed problems in a Hilbert space setting where instead of exact data y noisy data $y^{\delta} \in X$ is give satisfying $\left\|y - y^{\delta}\right\| \leq \delta$ with known noise level δ. Assuming the unknown solution belongs to some general source set M we propose a class of regularization methods that lead to optimal error bounds on the set M toward the minimal-norm, least-squares solution of an ill-posed linear operator equation in the presence of noisy data. Our results cover the special case of finitely smoothing operators A and extend recent results for infinitely smoothing operators. In addition, we generalize our results to the method of iterative regularization of order m.

Keywords

Global optimization, Ill-posed problems, A priori parameter choice, A poste- riori rule, General source conditions.

1. Introduction

Many works have been done for regularization of linear ill-posed problems [1-6]. We are concerned with the problem of determining solutions $x^{\dagger}$ for the linear ill-posed problems

$$Ax = y, \quad \left(y \in R\left(A\right)\right) \tag{1}$$

where A is a bounded non-negative, self-adjoint and injective operator on a Hilbert space δ and $y \in R\left(A\right)$, the range of A. This problem is in general ill-posed in the sense that even if a unique solution for (1) exists, the solution may not depend continuously on the data y. This situation occurs if $R\left(A\right)$ is not closed. For each $\delta > 0$, let $y^{\delta} \in X$ such that

$$\left\|y - y^{\delta}\right\| \leq \delta \tag{2}$$

and known noise level δ.

The problem of solving (1) is, in general, ill-posed. By ill-posedness, we always mean that the solutions do not depend continuously on the data. In the case of multiple solutions this is understood in the sense of multivalued mappings. To cope with the ill-posedness, problem (1) has to be regularized. A well-known and effective technique is Tikhonov regularization. In this method a solution of problem (1) is approximated by a solution of the minimization problem [7-8]

$$\min_{x \in X}\left\{\left\|Ax - y^{\delta}\right\|^{2} + \alpha\left\|x\right\|^{2}\right\} \tag{3}$$

where α is regarded as the regularized parameter, x_{α}^{δ} is the Tikhonov regularization solution.

In the case of non-negative and self-adjoint operators A the least squares minimization in problem (3), equivalently

$$A^{*}Ax + \alpha x = A^{*}y^{\delta} \tag{4}$$

can be replaced by the general regularized equation

$$x = q_{\alpha}\left(A\right) y^{\delta} \tag{5}$$

This method we consider to compute the regularized approximation x_{α}^{δ} by solving (5) is toextend some results from the results [9-13].

This paper is organized as follows. In Section 2 we make preliminary work for the approximate solution of problem (1) with data y^{δ}. In Section 3 and 4 we prove that the regularization method we propose is quasi-order optimal on some general source set M provided the regularization parameter is chosen either a priori or a posteriori by the rule of Engl. In Section 5 we generalize our results to the proposed method of iterative regularization of order m.

2. Optimality and quasi-order optimality

For the stable approximate solution of problem (1) some regularization technique has to be applied, which provides regularized approximations $x_{\alpha}^{\delta} = R_{\alpha}^{\delta}y^{\delta}$ with the property $x_{\alpha}^{\delta} \to x^{\dagger}$ as $\delta \to 0$ where the regularization parameter $\alpha=\alpha\left(\delta, y^{\delta}\right)$ has to be chosen properly. Hence, regularizedapproximations x_{α}^{δ} depend continuously on the data [14-18].

In order to guarantee certain convergence rates for $\left\|x_{\alpha}^{\delta}-x^{\dagger}\right\|$, the set of solutions of problem (1) has to be restricted to certain source sets. In this paper we are interested in quasi-order optimality results under general source conditions of the type $x^{\dagger} \in M_{\varphi,E}$ with

$$M_{\varphi,E} = \left\{x \in X \,\middle|\, x = \varphi\left(A\right)\upsilon, \left\|\upsilon\right\| \le E\right\} \tag{6}$$

and source functions φ satisfying

Assumption A1. φ: $\left(0, a\right] \to \left(0, +\infty\right)$ with $\left\|A\right\| \le a$ is continuous and possesses the following properties:

(i) φ is strictly monotonically increasing on $\left(0, a\right]$ with $\lim_{\mu\to 0}\varphi\left(\mu\right) = 0$.

(ii) Thefunction g: $\left(0, \varphi^{2}\left(a\right)\right] \to \left(0, a^{2}\varphi^{2}\left(a\right)\right)$, implicitly defined by $g\left(\varphi^{2}\left(\mu\right)\right)=\lambda^{2}\varphi^{2}\left(\mu\right)$, isconvex.

In (6), the operator function φ is defined via spectral representation:

$$\varphi\left(A\right)=\int_{0}^{a}\varphi\left(\mu\right)dE_{\mu} \qquad \left(\left\|A\right\| \le a\right) \tag{7}$$

where $A=\int_{0}^{a}\mu dE_{\mu}$ is the spectral representation and $\left\{E_{\mu}\right\}_{0\le\mu\le a}$ the spectral family of A. We may observe that, since A is assumed to be self-adjoint and non-negative, $\sigma\left(A\right) \subseteq \left[0, a\right]$, where $\sigma\left(A\right)$ denotes the spectrum of the operator A.

It is well-known that any operator $R : X \to X$ can be considered as a special method for solving linear ill-posed problems such as (1). Then the approximate solution is given by Ry^{δ}. Let us consider the worst case error $\Delta\left(\delta, R\right)$ for identifying the solution $x^{\dagger}$ of problem (1) from $y^{\delta} \in X$ satisfying (2) and $x^{\dagger} \in M_{\varphi,E}$ which is defined by

$$\Delta\left(\delta, R\right) = \sup\left\{\left\|Ry^{\delta} - x^{\dagger}\right\| \middle| x^{\dagger} \in M_{\varphi,E}, y^{\delta} \in X, \left\|y - y^{\delta}\right\| \leq \delta\right\} \tag{8}$$

This worst case error characterizes the maximal error of the method R if the solution $x^{\dagger}$ of problem (1) varies in the set $M_{\varphi,E}$. An optimal method R_{opt} is characterized by $\Delta\left(\delta, R_{opt}\right) = \inf_{R} \Delta\left(\delta, R\right)$. It can easily be realized that $\inf\limits_{R} \Delta\left(\delta, R\right) \geq \omega\left(\delta, M_{\varphi,E}\right)$

with

$$\omega\left(\delta, M_{\varphi,E}\right) = \sup\left\{\left\|x\right\| \middle| x \in M_{\varphi,E}, \left\|Ax\right\| \leq \delta\right\} \tag{9}$$

For estimating the modulus of continuity $\omega\left(\delta, M_{\varphi,E}\right)$ of the inverse operator A^{-1} on the source set $M_{\varphi,E}$, we introduce the function $\rho : \left(0, \varphi\left(a\right)\right] \to \left(0, a\varphi\left(a\right)\right]$, defined implicitly by $\rho\left(\varphi\left(\mu\right)\right) = \mu\varphi\left(\mu\right)$, or explicitly by

$$\rho\left(\mu\right) = \mu\varphi^{-1}\left(\mu\right) \tag{10}$$

Lemma 2.1. Let $M_{\varphi,E}$ be given by (6) and let Assumption A1 be satisfied. If δ is sufficiently small such that $\frac{\delta}{E} \leq a\varphi\left(a\right)$, then

$$\omega\left(\delta, M_{\varphi,E}\right) \leq E\rho^{-1}\left(\frac{\delta}{E}\right) \tag{11}$$

with ρ given by (10). If $\frac{\delta}{E} \in \sigma\left(A\varphi\left(A\right)\right)$, then there holds the equality in (11).

Proof. From (6) and (9) we have

$$\omega\left(\delta, M_{\varphi,E}\right) = \sup\left\{\left\|\varphi\left(A\right)\upsilon\right\| \middle| \left\|A\varphi\left(A\right)\upsilon\right\| \leq \delta, \left\|\upsilon\right\| \leq E\right\} \tag{12}$$

Substituting $\varphi\left(A\right)\upsilon = E\omega$ provides

$$\omega\left(\delta, M_{\varphi,E}\right) = E\sup\left\{\left\|\varphi\left(A\right)\upsilon\right\| \middle| \left\|A\omega\right\| \leq \frac{\delta}{E}, \left\|\left[\varphi\left(A\right)\right]^{-1}\omega\right\| \leq 1\right\} \tag{13}$$

Now choose ω in the range of $\varphi\left(A\right)$ such that the side conditions $\left\|A\omega\right\| \leq \frac{\delta}{E}$ and $\left\|\left[\varphi\left(A\right)\right]^{-1}\omega\right\| \leq 1$ of (13) are satisfied. Since the function g of Assumption A1(ii) is convex, by exploiting Jensens inequality we obtain

$$
\begin{aligned}
g\left(\frac{\|\omega\|^2}{\left\|\left[\varphi(A)\right]^2\right\|}\right) &= g\left(\frac{\int_0^a \varphi^2(\mu)\left[\varphi(\mu)\right]^{-2} d\left\|E_\mu\omega\right\|^2}{\int_0^a \left[\varphi(\mu)\right]^{-2} d\left\|E_\mu\omega\right\|^2}\right) \\
&\le \frac{\int_0^a g\left(\varphi^2(\mu)\right)\left[\varphi(\mu)\right]^{-2} d\left\|E_\mu\omega\right\|^2}{\int_0^a \left[\varphi(\mu)\right]^{-2} d\left\|E_\mu\omega\right\|^2} \qquad (14) \\
&= \frac{\|A\omega\|^2}{\left\|\left[\varphi(A)\right]^{-1}\omega\right\|^2}
\end{aligned}
$$

We exploit the side condition $\left\|\left[\varphi(A)\right]^{-1}\omega\right\| \le 1$, and take into consideration that the function defined in Assumption A1 (ii) possesses the explicit form

$$g(\varphi) = \mu\left[\varphi^{-1}\left(\sqrt{\mu}\right)\right]^2 \qquad (15)$$

exploit the monotonicity of φ^{-1} as well as inequality (14) and obtain

$$
\begin{aligned}
\left[\varphi^{-1}\left(\|\omega\|\right)\right]^2 &= \left[\varphi^{-1}\left(\frac{\|\omega\|}{\left\|\left[\varphi(A)\right]^{-1}\omega\right\|}\right)\right]^2 \\
&= \frac{\left\|\left[\varphi(A)\right]^{-1}\omega\right\|^2}{\|\omega\|^2}\, g\left(\frac{\|\omega\|^2}{\left\|\left[\varphi(A)\right]^{-1}\omega\right\|^2}\right) \qquad (16) \\
&\le \frac{\|A\omega\|^2}{\|\omega\|^2}
\end{aligned}
$$

This inequality attains the form $\rho\left(\|\omega\|\right) \le \|A\omega\|$, giving $\|\omega\| \le \rho^{-1}\left(\|A\omega\|\right)$, where ρ is defined by (9). Due to the monotonicity of ρ^{-1} and the assumption $\frac{\delta}{E} \le a\varphi(a)$, we obtain $\|\omega\| \le \rho^{-1}\left(\|A\omega\|\right) \le \rho^{-1}\left(\frac{\delta}{E}\right)$. From this estimate and (13) we obtain (11).

In the second part we prove that in (11) equality holds provided $\frac{\delta}{E} \in \sigma A\varphi(A)$. Assume that $\frac{\delta}{E}$ is an eigenvalue of the operator $A\varphi(A)$ and υ_0 is a corresponding eigenvectorwith $\|\upsilon_0\| = E$. Then $A\varphi(A)\upsilon_0 = \frac{\delta}{E}\upsilon_0$, consequently, $\left\|A\varphi(A)\upsilon_0\right\| = \delta$. Hence, in view of (12) we conclude that $\omega\left(\delta, M_{\varphi,E}\right) \ge \left\|\varphi(A)\upsilon_0\right\|$.

From $\rho\left(\varphi\left(A\right)\right)\upsilon_0 = A\varphi\left(A\right)\upsilon_0 = \frac{\delta}{E}\upsilon_0$ we obtain $\varphi\left(A\right)\upsilon_0 = \rho^{-1}\left(\frac{\delta}{E}\right)\upsilon_0$, consequently, $\omega\left(\delta, M_{\varphi,E}\right) \geq E\rho^{-1}\left(\frac{\delta}{E}\right)$. Hence, due to (11) we have

$$\omega\left(\delta, M_{\varphi,E}\right) = E\rho^{-1}\left(\frac{\delta}{E}\right) \tag{17}$$

If $\frac{\delta}{E} \in \sigma\left(A\varphi\left(A\right)\right)$ is not an eigenvalue, then $\frac{\delta}{E}$ belongs to the approximate eigenspectrum of $A\varphi\left(A\right)$ as $A\varphi\left(A\right)$ is self-adjoint, and in that case, the proof of (17) followswith small modifications [19-22].

For the proof of Lemma 2.1 in the case of compact operators A see the reference (e.g. Ivanov et al., 1969) . Our proof is based on the proof of Lemma 2.1 in the reference (e.g. Tautenhahn, 1998) and it is more general since the operator A is not necessarily compact. Note that estimate (11) can also be given in terms of the function g defined in Assumption A1(ii) and possessesthe equivalent form $\omega\left(\delta, M_{\varphi,E}\right) \leq E\sqrt{g^{-1}\left(\frac{\delta^2}{E^2}\right)}$.

Due to Lemma 2.1 the following definition makes sense.

Definition2.2. Let Assumption A1 be satisfied and ρ be given by(10). Then, anyregularization method R_α^δ, or any regularized approximation $x_\alpha^\delta = R_\alpha^\delta y^\delta$ for for problem (1),(2) is called

(i) optimal on the set $M_{\varphi,E}$ if $\left\|x_\alpha^\delta - x^\dagger\right\| \leq E\rho^{-1}\left(\frac{\delta}{E}\right)$.

(ii) quasi-order optimal on the set $M_{\varphi,E}$ if $\left\|x_\alpha^\delta - x^\dagger\right\| \leq cE\rho^{-1}\left(\frac{\delta}{E}\right)$, where c is dependent of α and δ.

3. A priori parameter choice

It is well known that in Lavrentiev regularization method the regularized approximation R_α^δ can be represented in the following form

$$x_\alpha^\delta = \left(A+\alpha I\right)^{-1} y^\delta \tag{18}$$

In this section we consider the general form $x_\alpha^\delta = q\left(\alpha, A\right) y^\delta = q_\alpha\left(A\right) y^\delta$, which is a generalization to the Larrentiev regularization approximation.

AssumptionA2. $q_\alpha\left(\mu\right) : \left(0, +\infty\right) \times \left[0, \|A\|\right] \to R^+, \|A\| \leq a$ is continuous and possesses thefollowing properties:

(i) $0 < \mu q_\alpha\left(\mu\right) \leq 1$ for $\alpha > 0$ and all $\mu \in \left(0, \|A\|\right]$.

(ii) There exists $c\left(\alpha\right) > 0$ such that $\mu - q_\alpha\left(\mu\right)\mu^2 \leq c\left(\alpha\right)$, where $\mu \in \left(0, \|A\|\right]$, and $c\left(\alpha\right)$ is strictlymonotonically on $\left(0, +\infty\right)$.

(iii) $\lim\limits_{a\to 0} q_\alpha\left(\mu\right) = \frac{1}{\mu}$ for $\mu \in \left(0, \|A\|\right]$.

We use the following notations for convenience:

$$x_\alpha = q(\alpha, A)\, y = q_\alpha(A)\, y \tag{19}$$

$$x_\alpha^\delta = q(\alpha, A)\, y^\delta = q_\alpha(A)\, y^\delta \tag{20}$$

which is the regularized approximation with exact data, the available noisy data, respectively.

Theorem 3.1. Let $M_{\varphi,E}$ be given by (6), $x^\dagger \in M_{\varphi,E}$, and Assumption A1, A2 be satisfied and let ρ be given by (10). Let x_α be the regularized approximation defined in (19) and let α be chosen a priori by

$$\alpha = c^{-1}\left(\varphi^{-1}\left(\rho^{-1}\left(\frac{\delta}{E}\right)\right)\right) \tag{21}$$

If the function φ is concave, then

$$\left\| x_\alpha - x^\dagger \right\| \le E\rho^{-1}\left(\frac{\delta}{E}\right) \tag{22}$$

Proof. Since $\lim_{\mu \to 0} \varphi(\mu) = 0$ and is concave we have $t\varphi(\mu) \le \varphi(t\mu)$ for $t \in [0,1]$. Choosing $t = \mu q_\alpha(\mu)$ and exploiting the monotonicity of φ provides

$$\left(1 - q_\alpha(\mu)\mu\right)\varphi(\mu) \le \varphi\left[\left(1 - q_\alpha(\mu)\mu\right)\mu\right] \le \varphi(c(\alpha)) \tag{23}$$

Let us use the notation $B_\alpha = I - q_\alpha(A)A$. Then, the regularization error can be expressed by

$$\begin{aligned} \left\| x_\alpha - x^\dagger \right\| &= \left\| \left(I - q_\alpha(A)A\right) x^\dagger \right\| \\ &\le E \sup \left| \left(q_\alpha(\mu)\mu - 1\right)\varphi(\mu) \right| \\ &\le E\varphi(c(\alpha)) \end{aligned} \tag{24}$$

For the regularization parameter α chosen by (21) there holds $\varphi(c(\alpha)) = \rho^{-1}\left(\frac{\delta}{E}\right)$, consequently, (22) follows from (24).

Theorem 3.2.Let $M_{\varphi,E}$ be given by (6), $x^\dagger \in M_{\varphi,E}$, and Assumption A1, A2 be satisfiedand let ρ be given by (10). Let x_α^δ be the regularized approximation defined in (20) and let α be chosen a priori by (21). If the function φ is concave, then

$$\left\| x_\alpha^\delta - x^\dagger \right\| \le 2E\rho^{-1}\left(\frac{\delta}{E}\right) \tag{25}$$

Proof. Due to $\rho(\mu) = \mu\varphi^{-1}(\mu)$ we obtain for $\mu = \rho^{-1}\left(\frac{\delta}{E}\right)$ the equation $\frac{\delta}{E} = \rho^{-1}\left(\frac{\delta}{E}\right)\varphi^{-1}\left(\rho^{-1}\left(\frac{\delta}{E}\right)\right)$. Hence, for α chosen by (21) we obtain

$$\frac{\delta}{E} = c(\alpha)\,\rho^{-1}\left(\frac{\delta}{E}\right) \tag{26}$$

Let x_α be given by (19). Then, Exploiting (25) and Assumption A2, we obtain

$$\left\|x_\alpha^\delta - x_\alpha\right\| \le \left\|q_\alpha(A)\left(y^\delta - y\right)\right\| \le \frac{\delta}{c(\alpha)} = E\rho^{-1}\left(\frac{\delta}{E}\right) \tag{27}$$

Now using the property of the triangle inequality (22) and (27), then holds the assertion.

4. A posteriori parameter choice

In Section 3 we have proved that the proposed regularization method provides quasi-order optimal error bounds (25) on the general set $M_{\varphi,E}$ given by (6) provided the regularizationparameter α is chosen a priori according to formula (21). Unfortunately, this a priori parameter choice requires the knowledge of the function φ, which is generally unknown. One prominent a posteriori rule for choosing α which does not require to know the function φ is Morozovs discrepancy principle (e.g. Morozov, 1966; Nair, 1999) in which α is chosen as the solution of the nonlinear scalar equation $\left\|Ax_\alpha^\delta - y^\delta\right\| = C\delta$ with some constant $C \ge 1$. Although Morozovss discrepancy principle works well for the method of Tikhonov regularization (3), it appears to be divergent for the method of Lavrentiev regularization.

In this section we discuss the rule of Engl for choosing the regularization parameter. This posteriori rule does not require to know the function φ which characterizes the set $M_{\varphi,E}$ given by (6). This rule reads as follows:

Rule of Engl. For each $p, q > 0, \delta > 0$, and y^δ fulfilling (2), there is a unique $\alpha > 0$ such that

$$d(\alpha) := \left\|Ax_\alpha^\delta - y^\delta\right\| = \frac{\delta^p}{\alpha^q} \tag{28}$$

In our first proposition, we estimate the regularization error $\left\|x_\alpha - x^\dagger\right\|$ where x_α is the regularized approximation with exact data, that is, x_α is given by (19).

Proposition 4.1. Let $x^\dagger \in M_{\varphi,E}$ with $M_{\varphi,E}$ given by (6), Assumption A1, A2 be satisfied and let ρ be given by (10). Let x_α the regularized approximation defined in (19) and let α be chosen by rule (28). If the function φ is concave, then

$$\left\|x_\alpha - x^\dagger\right\| \le \left(\frac{\delta^{p-1}}{\alpha^q} + 1\right) E\rho^{-1}\left(\frac{\delta}{E}\right) \tag{29}$$

Proof. Notice that φ is a concave function with $\lim_{\mu\to 0}\varphi(\mu) = 0$ we have $t\varphi(\mu) \le \varphi(t\mu) \quad t \in [0,1]$.

We multiply by $t^2\varphi^2(\mu)$ and obtain

$$t^2\varphi^2(\mu)\left[\varphi^{-1}\left(t\varphi(\mu)\right)\right]^2 \le t^4\mu^2\varphi^2(\mu) \quad t \in [0,1] \tag{30}$$

Therefore, we obtain $g\left(t^2\varphi^2(\mu)\right) \le t^4\mu^2\varphi^2(\mu)$.

Choosing $t = 1 - q_\alpha(\mu)\mu$ yields

$$g\left[\left(1 - q_\alpha(\mu)\mu\right)^2 \varphi^2(\mu)\right] \le \mu^2\left(1 - q_\alpha(\mu)\mu\right)^4 \varphi^2(\mu) \tag{31}$$

Let α be the regularization parameter chosen by rule (28).

Notice $d(\alpha) = \left\|\left(I - Aq_\alpha(A)\right) y^\delta\right\|$ we obtain

$$\begin{aligned}\left\|\left(I - Aq_\alpha(A)\right) y\right\| &\leq \left\|\left(I - Aq_\alpha(A)\right) y^\delta\right\| + \left\|\left(I - Aq_\alpha(A)\right)\left(y^\delta - y\right)\right\| \\ &\leq \delta + \frac{\delta^p}{\alpha^q}\end{aligned} \tag{32}$$

Recall that since $x^\dagger \in M_{\varphi,E}$, $x^\dagger = \varphi(A)\upsilon$ for some $\upsilon \in X$ with $\|\upsilon\| \leq E$, so that we obtain $\left(I - Aq_\alpha(A)\right) x^\dagger = \left(I - Aq_\alpha(A)\right)\varphi(A)\upsilon$.

Using Assumption A1, A2 as well as (31) and (32), we obtain

$$\begin{aligned} g\left(\frac{\left\|\left(I - Aq_\alpha(A)\right) x^\dagger\right\|^2}{\|\upsilon\|^2}\right) &= g\left(\frac{\int_0^a \left(1 - q_\alpha(\mu)\mu\right)^2 \varphi^2(\mu)\, d\left\|E_\mu \upsilon\right\|^2}{\int_0^a d\left\|E_\mu \upsilon\right\|^2}\right) \\ &\leq \frac{\int_0^a g\left(\left(1 - q_\alpha(\mu)\mu\right)^2 \varphi^2(\mu)\right) d\left\|E_\mu \upsilon\right\|^2}{\int_0^a d\left\|E_\mu \upsilon\right\|^2} \\ &\leq \frac{\int_0^a \mu^2\left(1 - q_\alpha(\mu)\mu\right)^4 \varphi^2(\mu)\, d\left\|E_\mu \upsilon\right\|^2}{\int_0^a d\left\|E_\mu \upsilon\right\|^2} \\ &= \frac{\left\|\left(I - Aq_\alpha(A)\right)^2 Ax^\dagger\right\|^2}{\|\upsilon\|^2} \\ &\leq \frac{\delta + \frac{\delta^p}{\alpha^q}}{\|\upsilon\|^2} \end{aligned} \tag{33}$$

Exploiting the monotonicity of φ^{-1} as well as (15) and (33) we obtain

$$\begin{aligned} \left[\varphi^{-1}\left(\frac{\left\|\left(I - Aq_\alpha(A)\right) x^\dagger\right\|}{\left(\frac{\delta^{p-1}}{\alpha^q} + 1\right)E}\right)\right]^2 &\leq \left[\varphi^{-1}\left(\frac{\left\|\left(I - Aq_\alpha(A)\right) x^\dagger\right\|}{\|\upsilon\|}\right)\right]^2 \\ &= \frac{\|\upsilon\|^2}{\left\|\left(I - Aq_\alpha(A)\right) x^\dagger\right\|^2}\, g\left(\frac{\left\|\left(I - Aq_\alpha(A)\right) x^\dagger\right\|^2}{\|\upsilon\|^2}\right) \\ &\leq \frac{\|\upsilon\|^2}{\left\|\left(I - Aq_\alpha(A)\right) x^\dagger\right\|^2} \frac{\left(\delta + \frac{\delta^p}{\alpha^q}\right)^2}{\|\upsilon\|^2} \\ &= \frac{\left(\delta + \frac{\delta^p}{\alpha^q}\right)^2}{\left\|\left(I - Aq_\alpha(A)\right) x^\dagger\right\|^2} \end{aligned} \tag{34}$$

By virtue of the definition of ρ according to $\rho(\mu)=\mu\varphi^{-1}(\mu)$, which together with the above estimate gives

$$\rho\left(\frac{\left\|\left(I-Aq_{\alpha}(A)\right)x^{\dagger}\right\|}{\left(\frac{\delta^{p-1}}{\alpha^{q}}+1\right)E}\right)\leq\frac{\delta}{E}. \tag{35}$$

Exploiting this estimate and the identity $\left\|x_{\alpha}-x^{\dagger}\right\|=\left\|\left(I-q_{\alpha}(A)A\right)x^{\dagger}\right\|$, we obtain (29).

Proposition 4.2.Let $M_{\varphi,E}$ be given by (6), $x^{\dagger}\in M_{\varphi,E}$, and Assumption A1, A2 be satisfied and let ρ be given by (10). Let α be chosen by rule (28). If the function φ is concave, then

$$\left(\frac{\delta^{p-1}}{\alpha^{q}}-1\right)\delta\leq Ec(\alpha)\varphi\left(c(\alpha)\right) \tag{36}$$

Proof. Exploiting the rule (28) we obtain

$$\frac{\delta^{p}}{\alpha^{q}}=\left\|\left(I-Aq_{\alpha}(A)\right)y^{\delta}\right\|\leq\delta+\left\|\left(I-Aq_{\alpha}(A)\right)y\right\| \tag{37}$$

Using the estimate as well as Assumption A2 and

$$\left[1-q_{\alpha}(\mu)\right]\mu\leq c(\alpha) \tag{38}$$

we obtain

$$\frac{\delta^{p}}{\alpha^{q}}-\delta\leq\left\|\left(I-Aq_{\alpha}(A)\right)y\right\|=\left\|\left(I-Aq_{\alpha}(A)\right)Ax^{\dagger}\right\|\leq Ec(\alpha)\varphi\left(c(\alpha)\right) \tag{39}$$

Now the assertion can be proved easily.

Theorem 4.3. Let $x^{\dagger}\in M_{\varphi,E}$ with $M_{\varphi,E}$ given by (6), let Assumption A1, A2 be satisfied and let ρ be given by (10). Let x_{α}^{δ} the regularized approximation defined in (19) and let α be chosen by rule (28). If the function φ is concave, then x_{α}^{δ} is quasi-order optimal on the set $M_{\varphi,E}$. In fact,

$$\left\|x_{\alpha}^{\delta}-x^{\dagger}\right\|\leq c_{0}E\rho^{-1}\left(\frac{\delta}{E}\right) \tag{40}$$

with

$$c_{0}=\frac{\delta^{2p-2}+\alpha^{2q}}{\alpha^{q}\left(\delta^{p-1}-\alpha^{q}\right)}\text{ for }1<\frac{\delta^{p-1}}{\alpha^{q}}\leq 2 \tag{41}$$

and

$$c_{0}=\frac{\delta^{p-1}+3\alpha^{q}}{\alpha^{q}}\text{ for }\frac{\delta^{p-1}}{\alpha^{q}}\geq 2 \tag{42}$$

Proof. By (34) and the monotonicity of φ there holds

$$\varphi^{-1}\left[\left(\frac{\delta^{p-1}}{\alpha^{q}}-1\right)\frac{\delta}{Ec(\alpha)}\right]\leq c(\alpha) \tag{43}$$

Exploiting $\rho(\mu)=\mu\varphi^{-1}(\mu)$ we conclude that

$$\rho\left[\left(\frac{\delta^{p-1}}{\alpha^{q}}-1\right)\frac{\delta}{Ec(\alpha)}\right]\leq\left(\frac{\delta^{p-1}}{\alpha^{q}}-1\right)\frac{\delta}{E} \tag{44}$$

Using the monotonicity of ρ^{-1}, we can conclude that

$$\frac{\delta}{c(\alpha)} \leq \frac{E\alpha^q}{\delta^{p-1}-\alpha^q}\rho^{-1}\left(\frac{\delta}{E}\right) \text{for } 1 < \frac{\delta^{p-1}}{\alpha^q} \leq 2 \tag{45}$$

In the case $\frac{\delta^{p-1}}{\alpha^q} \geq 2$, we use the monotonicity of φ^{-1} and obtain from (36) the estimate

$$\varphi^{-1}\left(\frac{\delta}{Ec(\alpha)}\right) \leq c(\alpha) \tag{46}$$

and instead of (37) the estimate

$$\frac{\delta}{c(\alpha)} \leq E\rho^{-1}\left(\frac{\delta}{E}\right) \text{ for } \frac{\delta^{p-1}}{\alpha^q} \geq 2 \tag{47}$$

Using Assumption A2 as well as (19) and (20) we obtain

$$\left\|x_\alpha^\delta - x_\alpha\right\| \leq \frac{2\delta}{c(\alpha)} \tag{48}$$

Therefore, by virtue of (37) and (38),

$$\left\|x_\alpha^\delta - x_\alpha\right\| \leq cE\rho^{-1}\left(\frac{\delta}{E}\right) \tag{49}$$

with

$$c = \frac{2\alpha^q}{\delta^{p-1} - \alpha^q} \text{ for } 1 < \frac{\delta^{p-1}}{\alpha^q} \leq 2 \tag{50}$$

and

$$c = 2 \text{ for } \frac{\delta^{p-1}}{\alpha^q} \geq 2 \tag{51}$$

Now the quasi-order optimal error bound (35) follows from (29) and (39).

5. Iterated regularization

In this section we are going to generalize our results of Sections 3 and 4 for the method ofproposed method for iterated regularization.Starting with $x_{\alpha,0}^\delta = 0$, in this method the regularized approximation $x_\alpha^\delta := x_{\alpha,m}^\delta$ is defined recursively by solving the m operator equations

$$x_{\alpha,k}^\delta = q_\alpha(A)\left(y^\delta + \alpha x_{\alpha,k-1}^\delta\right) \quad (k = 1, 2, \cdots, m) \tag{52}$$

In the case of exact data y, we define $x_\alpha := x_{\alpha,m}$ recursively by solving the m operator equations

$$x_{\alpha,k} = q_\alpha(A)\left(y + \alpha x_{\alpha,k-1}\right) \quad (k = 1, 2, \cdots, m) \tag{53}$$

Now we conclude that $x_\alpha^\delta = g_\alpha(A)y^\delta$ and $x_\alpha = g_\alpha(A)y$ where

$$g_\alpha(\mu) = \frac{q_\alpha(\mu)}{1-\alpha q_\alpha(\mu)}\left[1-\left(\alpha q_\alpha(\mu)\right)^m\right] \quad \left(0 \leq \mu \leq \|A\|\right) \tag{54}$$

Theorem 5.1. Let $M_{\varphi,E}$ given by (6), $x^\dagger \in M_{\varphi,E}$, Assumption A1, A2 be satisfied and let ρ be given by (10). Let $x_\alpha^\delta := x_{\alpha,m}^\delta$ the regularized approximation defined in (40) and let α chosen a priori by (20). If the function φ is concave, then

$$\left\| x_\alpha^\delta - x^\dagger \right\| \le (m+1) E \rho^{-1}\left(\frac{\delta}{E}\right) \tag{55}$$

Proof. Let us use the notations.

$$D_\alpha = \frac{I - (\alpha + A) q_\alpha(A) + q_\alpha(A) A \left(\alpha q_\alpha(A)\right)^m}{I - q_\alpha(A)\alpha} \tag{56}$$

$$r_\alpha(\mu) = \frac{1 - (\alpha + \mu) q_\alpha(\mu) + q_\alpha(\mu)\mu\left(\alpha q_\alpha(\mu)\right)^m}{1 - q_\alpha(\mu)\alpha} \tag{57}$$

Since the function φ is concave , we have

$$t\varphi(\mu) \le \varphi(t\mu) \text{ for } 0 < t \le 1 \tag{58}$$

We use this inequality with

$$t = \frac{1 - (\mu + \alpha) q_\alpha(\mu)}{1 - q_\alpha(\mu)\mu} \tag{59}$$

exploit the representation $x^\dagger - x_\alpha = D_\alpha x^\dagger$ and obtain due to $x^\dagger \in M_{\varphi,E}$, Assumption A2 and the monotonicity of φ that

$$\begin{aligned} \left\| x_\alpha - x^\dagger \right\| &\le E \sup \left| r_\alpha(\mu)\varphi(\mu) \right| \\ &\le \sup_{\mu\in(0,a]} \left| \frac{1 - (\alpha + \mu) q_\alpha(\mu)}{1 - \alpha q_\alpha(\mu)} \varphi(\mu) \right| \\ &\le E\varphi\left[\mu \frac{1 - (\alpha + \mu) q_\alpha(\mu)}{1 - \alpha q_\alpha(\mu)}\right] \\ &= E\varphi\left(c(\alpha)\right) \end{aligned} \tag{60}$$

We may conclude that $x_\alpha^\delta - x_\alpha = q_\alpha(A)\left(y^\delta - y\right)$ with g_α as in (42). Since

$$\begin{aligned} g_\alpha(\mu) &= \frac{q_\alpha(\mu)}{1 - \alpha q_\alpha(\mu)}\left[1 - \left(\alpha q_\alpha(\mu)\right)^m\right] \\ &\le \frac{q_\alpha(\mu)}{1 - \alpha q_\alpha(\mu)}\left[1 - 1 - m\left(\alpha q_\alpha(\mu) - 1\right)\right] \\ &= m q_\alpha(\mu) \\ &\le \frac{m}{c(\alpha)} \end{aligned} \tag{61}$$

We can obtain

$$\left\| x_\alpha^\delta - x_\alpha \right\| \le m \frac{\delta}{c(\alpha)} \tag{62}$$

For α chosen according to (21) we obtain

$$\varphi\left(c(\alpha)\right) = \rho^{-1}\left(\frac{\delta}{E}\right) \tag{63}$$

and

$$\frac{\delta}{c(\alpha)} = E\rho^{-1}\left(\frac{\delta}{E}\right) \tag{64}$$

Then by using (44) and (45) we derive the assertion.

Theorem 5.2. Let $M_{\varphi,E}$ given by (6), $x^{\dagger} \in M_{\varphi,E}$, Assumption A1, A2 be satisfied and let ρ be given by (10).Let $x_{\alpha}^{\delta} := x_{\alpha,m}^{\delta}$ the regularized approximation defined in (40) and let α bechosen by rule (28). If the function φ is concave, then

$$\left\|x_{\alpha}^{\delta} - x^{\dagger}\right\| \leq c_0 E\rho^{-1}\left(\frac{\delta}{E}\right) \tag{65}$$

with

$$c_0 = \frac{\delta^{2p-2} + (m-1)\alpha^{2q}}{\alpha^{q}\left(\delta^{p-1} - \alpha^{q}\right)} \text{ for } 1 < \frac{\delta^{p-1}}{\alpha^{q}} \leq 2 \tag{66}$$

and

$$c_0 = \frac{\delta^{p-1} + (m+1)\alpha^{2q}}{\alpha^{q}} \text{ for } \frac{\delta^{p-1}}{\alpha^{q}} \geq 2 \tag{67}$$

Proof. By virtue of $y^{\delta} - Ax_{\alpha}^{\delta} = D_{\alpha}y^{\delta}$,the function d in (28) can be written in theequivalent form

$$d(\alpha) = \left\|D_{\alpha}y^{\delta}\right\|$$

Therefore, for α chosen by rule (28) we have

$$\left\|D_{\alpha}y\right\| \leq \left\|D_{\alpha}y^{\delta}\right\| + \left\|D_{\alpha}\left(y - y^{\delta}\right)\right\| \leq \frac{\delta^{p}}{\alpha^{q}} + \delta \tag{68}$$

Exploiting (47), we obtain in analogy to the proof of Proposition 4.1 that α chosen by rule (28) we obtain

$$\left\|x_{\alpha} - x^{\dagger}\right\| \leq \left(\frac{\delta^{p-1}}{\alpha^{q}} + 1\right) E\rho^{-1}\left(\frac{\delta}{E}\right) \tag{69}$$

Now we follow the proof of Proposition 4.2 and obtain the relation (34) in this case as well. Finally we follow the proof of Theorem 4.3 and obtain the quasi-order optimal error bound (46).

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References

[1] B. J. Stable, Solutions of Inverse Problems, Braunschwelg, Vieweg, 1986.

[2] H. W. Engl, Discrepancy Principles for Tikhonov Regularization of Ill-Posed Problems Leading to Optimal Convergence Rates, Journal of optimization theory and applications. 52 (1987), 209-215.

[3] L.J. Wang, Y.X. Xie, Z.J. Wu, Y.X. Du, K.D. He, A new fast convergent iteration regularization method, Engineering with Computers 35 (2019) 127–138.

[4] L.J. Wang, L. Xu, Y.X. Xie, Y.X. Du, X. Han, A new hybrid conjugate gradient method for dynamic force reconstruction, Advances in Mechanical Engineering 11 (1) (2019) 1–21.

[5] L.J. Wang, J.W. Liu, Y.X. Xie, Y.T. Gu, A new regularization method for the dynamic load identification of stochastic structures, Computers and Mathematics with Applications 76 (2018) 741–759.

[6] L.J. Wang, Y.X. Xie, Q.C. Deng, The dynamic behaviors of a new impulsive predator prey model with impulsive control at different fixed moments, Kybernetika 54 (2018) 522-541.

[7] H. W. Engl, M. Hanke, A. Neubauer, Regularization of Inverse Problems, Kluwer, Dordrecht, 1996.

[8] C. W. Groetsch, J. Guacaneme, Arcangelis method for Fredholm equations of the first kind, Proceedings of the american mathematical society. 99 (1987), 256-260.

[9] B. Hofmann, Regularization of Applied Inverse and Ill-Posed Problems, Teubner, Leipzig, 1986.

[10] V. Ivanov, T. Korolyuk, Error estimates for solutions of ill-posed problems, Computers & math- ematics with applications. 9 (1969), 35-49.

[11] M. M. Lavrentiev, Some Improperly Posed Problems of Mathematical Physics, Springer-Verlag, New York, 1967.

[12] A. K. Louis, Inverse und Schlecht Gestellte Probleme, Teubner, Stuttgart, 1989.

[13] P. Mathe, S. V. Pereverzev, Geometry of linear ill-posed problems invariable Hilbert scales, Inverse Problems. 19 (2003), 789-803.

[14] V. A. Morozov, On the solution of functional equations by the method of regularization, Soviet math. Dokl. 7 (1966), 414-417.

[15] M. T. Nair, Functional Analysis: A First Course: Prentice-Hall of India, New Delhi, 2002.

[16] M. T. Nair, On Morozovs method for Tikhonov regularization as an optimal order yielding algorithm, Zeitschrift fur analysis und ihre anwendungen. 18 (1999), 37-46.

[17] M. T. Nair, E. Schock, U. Tautenhahn, Morozov discrepancy principle under general source conditions, Zeitschrift fur analysis und ihre anwendungen 22 (2003) 199-214.

[18] M. T. Nair, U. Tautenhahn, Lavrentiev Regularization for Linear Ill-Posed Problems under Gen- eral Source Conditions, Journal of mathematical analysis and applications. 23 (2004), 167-185.

[19] E. Schock, Approximate solution of ill-posed equations: arbitrary slow convergence vs. superconvergence. In: Constructive Methods for the Practical Treatment of Integral Equations (eds.:G. Hammerlin and K. H. Hofmann), Birkhauser Verlag, Basel, 1985.

[20] U. Tautenhahn, Optimality for ill-posed problems under general source conditions, Numerical functional analysis and optimization. 19 (1998), 377-398.

[21] G. M. Vainikko, A. Y. Veretennikov, Iteration Procedures in Ill-Posed Problems (in Russian), Nauka, Moscow, 1986.

[22] Z. Chen, C. H. Xiang, K. Q. Zhao, X. W. Liu, Convergence analysis of Tikhonov-type regularization algorithms for multiobjective optimization problems, Applied Mathematics and Computation 211 (2009) 167-172.