The Rigidity of the Poisson Diffeomorphisms on Some Poisson Manifolds

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Abstract

This paper studies the rigidity of the Poisson diffeomorphisms on some Poisson manifolds. Consider a sequence of Poisson diffeomorphisms, if the sequence converges in different topology, the property of the limit of this sequence is studied. Using the Poisson capacity method and symplectic leaf structure, we show the C^1 closed theorem and C^0 closed results in some conditions.

Keywords

Rigidity theory, Poisson bracket, Poisson diffeomorphism.

1. Introduction

We mainly consider the rigidity of the Poisson diffeomorphisms on the Poisson manifold in this paper. The group of diffeomorphisms is very important in the studying of symplectic and Poisson manifold, it establishes a bridge between geometry and dynamical systems. In symplectic geometry, the rigidity of symplectic diffeomorphisms is investigated deeply. By the definition of symplectic diffeomorphisms, if the sequence of symplectic diffeomorphisms converges in C^1topology, then the limit map is also a symplectic diffeomorphism. Suppose that the sequence of symplectic diffeomorphisms converges locally uniformly to a map, if the limit map is differentiable at the origin, then the differential of the map is symplectic. Symplectic diffeomorphism is volume preserving, but in C^0sense, a volume preserving map generally may not be approximated by symplectic maps [2, 5]. So the rigidity of the symplectic diffeomorphism in C^0sense becomes an interesting question. Eliashberg, Gromov, Ekeland, Hofer and other people made great contributions to this area [2, 3, 4, 5]. Eliashberg, Gromov proved the following results [3, 4, 5]:

Theorem 1. The group of symplectic diffeomorphisms of a compact symplectic manifold is C^0closed in the group of all diffeomorphisms.

Furthermore, consider the group of symplectic diffeomorphisms under the locally uniform limits, the following results hold:

Theorem 2[5]. Assume $h_j : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a sequence of homeomorphisms satisfying

$$c(h_j(E)) = c(E) \tag{1}$$

for all ellipsoids E. Assume that $h_j$converges locally uniformly to a homeomorphism h of $\mathbb{R}^{2n}$. Then

$$c(h_j(E)) = c(E) \tag{2}$$

holds for all ellipsoids E.

The Hamiltonian diffeomorphism is the time one map of the Hamiltonian flow, all Hamiltonian diffeomorphisms form a group, and we know that this group is a subgroup of the group of symplectic diffeomorphism. One natural question is that the C^0 rigidity theorem still holds for the Hamiltonian diffeomorphisms. Oh and Müller studied the C^0limits of the Hamiltonian diffeomorphisms and give the notations of Hamiltonian topology [11]. They consider both the limits of the Hamiltonian flows and Hamiltonian functions; the Hamiltonian topology involves the C^0 topology and the Hofer topology. The Hofer topology is induced by the bi-invariant Finsler metric which was first defined by Hofer [5, 6].

In this paper we want to investigate the rigidity phenomenon for Poisson diffeomorphisms on the Poisson manifold. The Poisson form may be degenerate and this brings difficulties to the research of Poisson rigidity, the symplectic methods cannot be used directly for general Poisson manifold. T. Rybicki studied the foliated, Poisson and Hamiltonian diffeomorphisms, defines the local flux homomorphism [14]. Inspired by the Poisson Hofer type metric which was defined by Sun and Zhang on the Poisson manifold [16], using the geometry structure of the Poisson manifold, we have the following results on some special Poisson manifold:

Theorem 3. The group of Poisson diffeomorphism of a C Poisson manifold is C^1closed in the group of all diffeomorphisms.

Consider the group of Poisson diffeomorphisms under the locally uniform limits, we have

Theorem 4. Assume $h_j : \mathbb{R}^n \to \mathbb{R}^n$ is a sequence of Hamiltonian diffeomorphisms on the standard Poisson space satisfying

$$c(h_j(E)) = c(E) \tag{3}$$

for all elliptic cylinder E. Assume that h_j converges locally uniformly to a Hamiltonian diffeomorphism h of the standard Poisson space. Then

$$c(h_j \tag{4}$$

holds for all elliptic cylinder E.

2. Preliminaries

In this section, we will introduce some notations needed in the proof; more results can be found in [1, 5, 9, 10, 14].

Definition 5[10]. For a smooth function H on the Poisson manifold, the Poisson vector is defined by

$$X_H[g] = \{g, H\} \tag{5}$$

for all smooth function g.

Definition 6[10]. We call a diffeomorphismϕPoisson diffeomorphism if it satisfies

$$\phi^*\{g,h\} = \{\phi^* g, \phi^* h\} \tag{6}$$

for all smooth function g, h.

Symplectic leaf is the connected component of the sets which can be connected by Hamiltonian path. Poisson manifold is the union of symplectic leaves. We call a Poisson manifold type C Poisson manifold if the Poisson manifold is compact, the leaves are dense and each leaf is compact, the Hamiltonian function on each leaf is well defined.

Definition 7. Let $l_\alpha \in M$be the symplectic leaf, $c_s$be the symplectic capacity, the Poisson capacity can be defined as following

$$c_p(M) = \sup_{l_\alpha} c_s(l_\alpha) \tag{7}$$

For arbitrary set A, and open subset U, we define the following function similarly

$$c \quad (A) = inf\{ c(U), A \subset U\} \tag{8}$$

3. Proof of main results

We now can give the proof of the main results in this part. Suppose that f_j,$g_j$are smooth functions on Poisson manifold, and $f_j \to f$, $g_j \to g$in the C^1topology, f,g are C^1functions, since the Poisson bracket is C^1in nature, the functions are somehow C^1closed.

Proof of Theorem 3. For Poisson diffeomorphismsϕ_j, if $\phi_j \to \phi$ in C^1topology, by the assumptions, the manifold is type C, so each leaf is compact. We know that the leaf carries a symplectic structure and coincides with the Poisson structure. Suppose that the restriction of the Poisson diffeomorphism on the symplectic leaf is ϕ_{j_L}, by theorem 1 we know that on each compact symplectic leaf, the diffeomorphisms satisfy

$$\phi_{j_L} \to \phi_L$$
$$\phi_{j_L}{}^*\{\mathrm{f},\mathrm{g}\}_L \to \phi_L{}^*\{\mathrm{f},\mathrm{g}\}_L \tag{9}$$

for any smooth function f,g. And hence

$$\phi_L{}^*\{\mathrm{f},\mathrm{g}\}_L = \{\phi_L{}^*\mathrm{f}, \phi_L{}^* g\}_L \tag{10}$$

Since L is arbitrary and dense, we have

$$\phi^*\{\mathrm{f},\mathrm{g}\} = \{\phi^*\mathrm{f}, \phi^* g\} \tag{11}$$

we finish the proof.

Remark 1. V. Humilière points the following convergence facts in the C^0topology [9]. Let f_j,$g_j$be smooth functions, and $f_j \to f$, $g_j \to g$in the C^0topology and

$$\{f_i, \mathrm{g}_i\} \quad \to h \tag{12}$$

But in general

$$\{\mathrm{f},\mathrm{g}\} \neq h \tag{13}$$

whenh=0, Cardin and Viterbo answered this question in some conditions [12]. Arnaud studied the rigidity for Tonelli Hamiltonians [8]. K. Samvelyan, F. Zapolsky studied the convergence under the L^ptopology [15].

Instead of studying the C^0rigidity directly, we study the convergence in the capacity sense.

Proof of Theorem 4: Recall the definition of standard Poisson space$(\mathbb{R}^n, \{\quad\}_0)$, If$n = 2m$, this is obvious by Theorem 2. When$n = 2m + 1$, the standard Poisson space can be viewed a generalization of symplectic space. Each symplectic leaf is a symplectic space with the same dimension; the symplectic form is the standard formω_0. The remaining one-dimensional subspace is perpendicular to each leaf.

Since the manifold is the standard Poisson space, we know that the Poisson capacity $c_p(M)$

is well defined, and satisfies the monotonicity and conformality. The following proof can be given by the similar way as in Theorem 6[5], we just give the outline of the proof. By the assumptions of the convergence:

$$h^{-1} \circ h_j \to id \tag{14}$$

The elliptic cylinder E is defined by the product of the ellipsoid and the remaining one-dimensional subspace in the standard Poisson manifold:

$$E = e \times \mathbb{R} \tag{15}$$

For εsmall enough and j large enough, we have

$$h^{-1} \circ h_j(1-\varepsilon)E \subset E \subset h^{-1} \circ h_j(1+\varepsilon)E \tag{16}$$
$$h_j(1-\varepsilon)E \subset h(E) \subset h_j(1+\varepsilon)E \tag{17}$$

By the conformality we have

$$(1-\varepsilon)^2 c(E) = c(h_j(1-\varepsilon)E)$$
$$(1+\varepsilon)^2 c(E) \geq c(h_j(1-\varepsilon)E) \tag{18}$$

So we have

$$c(E) = c(h(E)) \tag{19}$$

Remark 4. Hofer and Viterbo proved the following rigidity involving the Hofer metric [7, 13]. For Hamiltonian diffeomorphisms ψ_j ,ψ, and $\varphi \in C^0(\mathbb{R}^{2n}, \mathbb{R}^{2n})$. If the following conditions hold:

$$d(\psi_j, \psi) \to 0$$

$$\psi_j \to \varphi \text{ locally uniformly} \tag{20}$$

Then

$$\psi_j \to \varphi \tag{21}$$

Remark 5. We know that Poisson G manifold is closed and each leaf is closed, so Poisson G manifold is Poisson C manifold, and hence the Poisson diffeomorphisms of a G Poisson manifold is C^1closed in the group of all diffeomorphisms.

Corollary 6. For a special three dimensional Poisson G manifold, consider the three dimensional ball removing a small ball with the same center. Let the leaf be the 2 sphere, in this case the group of Poisson diffeomorphism is C^0closed in the group of all diffeomorphisms, that is if $\phi_j \to \phi$ in C^0topology, then ϕ is a Poisson diffeomorphism.

Proof: Since the leaf is dense and closed, we know that

$$\phi_{j_L} \to \phi_L \tag{22}$$

on each leaf, by C^0closed theorem of symplectic diffeomorphisms, we know that ϕ_L is symplectic diffeomorphism on each leaf, so we finish the proof.

Another example is the standard Poisson space$(\mathbb{R}^3, \{\quad\}_0)$. For other Poisson manifold, in order to use the symplectic methods, we should study the leaf structure carefully and there should be some more manifolds have these rigidity properties.

4. Conclusion

In this paper we study the rigidity of the Poisson diffeomorphisms on a class of special manifold named Poisson C manifold. We show the C^0 convergence and C^1 convergence of the Poisson diffeomorphisms. This partly generalizes the symplectic rigidity results to the Poisson manifolds.

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