Lyapunov-type Inequalities for Some Third-order Dynamic Equations on Time Scales

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Abstract

Inspired by some literatures, we establish some new Lyapunov-type inequalities for the third-order dynamic equation on time scales. Furthermore, our results are extension of the conclusions in literature.

Keywords

Lyapunov-type inequalities, Third-order dynamic equation, Time scale.

1. Introduction

Lyapunov [1] [2], who is a famous mathematician of Russia, first obtained the following conclusions in 1907 which will be introduced in the next section.

The inequality which obtained by Lyapunov in literature [2] is so-called Lyapunov inequality which plays an important role in studying the properties and applications of solution from some differential equations and difference equations. Because of the wide applications of the Lyapunov inequality, there are a lot of literatures about its extension and improvement. In 1970,the Lyapunov inequality was generalized to some second order nonlinear differential equations by Eliason [3]; in 1999, N. parhi and S. Pachpatte [1] applied the Lyapunov inequality to a third order differential equation. And lemma1.2 is the main contents of [1].Moreover, the Lyapunov inequality has been extended to even order differential equation by X.Yang[4],to odd order differential equation by X.Yang [5] in 2010.And Q.Zhang and etc.[6] generalized the Lyapunov inequality to even order difference equation and obtained relevant conclusions. For the limited length of the article, we only list several literature about our study. If you have interest in the other research and you could refer to [7]-[9].A time scale is an arbitrary nonempty closed subset of the real numbers. The theories of calculus on time scales is projected by Hilger[10],which unifies differential and difference. About the conclusions and symbols of time scale in this paper, readers could refer to [11].

For the convenience of research call all the improved and generalized Lyapunov inequalities as Lyapunov-type inequalities.

2. Preliminaries

There are some lemmas and definitions, which are introduced for the reader understand the next section.

Lemma 1.1[2] If $y(t)$ is a real solution of the equation

$$y''(t)+p(t)y(t)=0, \tag{2.1}$$

where $y(a)=y(b)=0(a<b), y(t)\neq 0, t\in(a,b)$ .Then we have the inequality

$$\int_a^b |p(t)|dt > \frac{4}{(b-a)}. \tag{2.2}$$

Lemma 1.2[1] If $y(t)$ is a real solution of the equation

$$y'''(t)+p(t)y(t)=0, \tag{2.3}$$

where $p(t)\in C([0,\infty]), y(a)=0=y(b)(0\leq a<b)$ and $y(t)\neq 0, t\in(a,b)$. Consider the following two cases:

Case1.There exists $d \in (a,b)$ such that $y^{"}(d) \neq 0$.

Case2. $y^{"}(d) \neq 0, t \in [a,b]$. In this case, three consecutive zero of $y(t)$ are considered, That is, $y(a) = y(b) = y(a^{'}) = 0 (0 \leq a < b < a^{'}), y^{"}(t) \neq 0$ for

$t \in (a,b), y^{"}(t) \neq 0$

$t \in (b,a^{'})$. And two results are as follows:

Case1. Then

$$\int_a^b | p(t) | dt > \frac{4}{(b-a)^2}, \tag{2.4}$$

Case2. Then

$$\int_a^{a^{'}} | p(t) | dt > \frac{4}{(a^{'}-a)^2}. \tag{2.5}$$

Definition 2.1 Let $y^{\Delta^{k+1}} = (y^{\Delta^k})^{\Delta}$ and $y^{\Delta^0} = y$ for $k \in N_0$, where $y(t)$ is a solution of the dynamic equation (3.1).

3. Main Results

In this paper, we use the similar methods in [1] to establish some new Lyapunov-type inequalities for following third-order dynamic equation

$$y^{\Delta^3}(t) + \sum_{k=1}^{n} p_k(t) y(t) = 0. \tag{3.1}$$

At first, we consider the Lyapunov-type inequalities on time scale $T = [a,b]$ for (3.1) with $y^{\Delta^2}(d) = 0, d \in [a,b]_T$ .Then we consider the Lyapunov-type inequalities on time scale $[a,a^{'}]_T = [a,b]_T \cup [b,a^{'}]$ for (3.1) with $y^{\Delta^2}(d) \neq 0$. Especially, our conclusions include some results of [1] as $T = R, n = 1$.

Dividing $[a,b]_T$ evenly into n parts, that is, $[a,b]_T = \cup_{i=1}^{n} T_i = \cup_{i=0}^{n-1} [t_i, t_{i+1}]$.

Theorem 3.1 Let $y(t)$ is a solution of (3.1) and $y(t_i) = 0, y(t) \neq 0 (t \in T, t \neq t_i)$, $t_0 = a, t_n = b (i = 0,1,\cdots,n)$ .If there exist $d \in T, d \neq t_i (i = 0,1,\cdots,n)$, then

$$\int_a^b \sum_{k=1}^{n} | p_k(t) | \Delta t > \frac{4n^2}{(b-a)^2}. \tag{3.2}$$

Proof. As the length of every sub time scale is same, Then

$\int_{t_{i-1}}^{t_i} | y^{\Delta}(t) | \Delta t = \int_{t_i}^{t_{i+1}} | y^{\Delta}(t) | \Delta t$ where $i = 1,2,\cdots,n-1$.

Let $M = \max_{t \in T} | y(t) | = | y(c) |, c \in T$, then, if $c \in T_1 = [a, t_1]$,we have

$$M = | y(c) | = | \int_a^c y^{\Delta}(t) \Delta t | \leq \int_a^c | y^{\Delta}(t) | \Delta t, \tag{3.3}$$

$$M = | y(c) | = | \int_c^{t_1} y^{\Delta}(t) \Delta t | \leq \int_c^{t_1} | y^{\Delta}(t) | \Delta t, \tag{3.4}$$

if $c \in T_2 = [t_1, t_2]$, we have

$$M = | y(c) | = | \int_{t_1}^c y^{\Delta}(t) \Delta t | \leq \int_{t_1}^c | y^{\Delta}(t) | \Delta t, \tag{3.5}$$

$$M = | y(c) | = | \int_c^{t_2} y^{\Delta}(t)\Delta t | \le \int_c^{t_2} | y^{\Delta}(t) | \Delta t, \tag{3.6}$$

if $c \in T_i = [t_i, t_{i+1}]$,we have

$$M = | y(c) | = | \int_{t_i}^{c} y^{\Delta}(t)\Delta t | \le \int_{t_i}^{c} | y^{\Delta}(t) | \Delta t, \tag{3.7}$$

$$M = | y(c) | = | \int_c^{t_{i+1}} y^{\Delta}(t)\Delta t | \le \int_c^{t_{i+1}} | y^{\Delta}(t) | \Delta t, \tag{3.8}$$

if $c \in T_n = [t_{n-1}, b]$,we have

$$M = | y(c) | = | \int_{t_{n-1}}^{c} y^{\Delta}(t)\Delta t | \le \int_{t_{n-1}}^{c} | y^{\Delta}(t) | \Delta t, \tag{3.9}$$

$$M = | y(c) | = | \int_c^{b} y^{\Delta}(t)\Delta t | \le \int_c^{b} | y^{\Delta}(t) | \Delta t. \tag{3.10}$$

Adding the left and right sides of the above $2n$ inequalities, then square of both sides. Hence, by applying the Cauchy-Schwarz inequality and the formula of integration by parts, we obtain

$$\begin{aligned}
4n^2M^2 &\le (\int_a^b | y^{\Delta}(t) | \Delta t)^2 \\
&\le (b-a)\int_a^b | y^{\Delta}(t) |^2 \Delta t \\
&= (b-a)\int_a^b y^{\Delta}(t)\Delta(y(t)) \\
&= (b-a)[(y(t)y^{\Delta}(t)) |_a^b - \int_a^b y^{\Delta^2}(t)y(\sigma(t))\Delta(t)] \\
&= -(b-a)\int_a^b y(\sigma(t))y^{\Delta^2}(t)\Delta t \\
&\le (b-a)\int_a^b | y(\sigma(t)) || y^{\Delta^2}(t) | \Delta t.
\end{aligned} \tag{3.11}$$

By definition2.1, we obtain

$$\begin{aligned}
| y^{\Delta^2}(t) | = | \int_d^t y^{\Delta^3}(t)\Delta t | &= | -\int_d^t \sum_{k=1}^{n} p_k(t)y(t)\Delta t | \\
&\le \int_d^t \sum_{k=1}^{n} | p_k(t) || y(t) | \Delta t \\
&\le \int_a^b \sum_{k=1}^{n} | p_k(t) || y(t) | \Delta t.
\end{aligned} \tag{3.12}$$

Substituting inequality (3.12) into (3.11), we have

$$\begin{aligned}
4n^2M^2 &\le (b-a)\int_a^b | y(\sigma(t)) || y^{\Delta^2}(t) | \Delta t \\
&\le (b-a)\int_a^b | y(\sigma(t)) | \Delta t \int_a^b \sum_{k=1}^{n} | p_k(t) || y(t) | \Delta t \\
&< (b-a)^2 M^2 \int_a^b \sum_{k=1}^{n} | p_k(t) | \Delta t.
\end{aligned} \tag{3.13}$$

which completes the proof.

Remark 3.1 $c \ge \sigma(a)$ .If and only if $T = Z$, the equal sign holds.

Proof. Let $a \le d < c$, it can be seen from the above conditions that $y^{\Delta}(c) = 0$ and $y(c) = \int_a^c y^{\Delta}(t)\Delta t$, then

$$
\begin{aligned}
(y(c))^2 &= (\int_a^c y^{\Delta}(t)\Delta t)^2 \\
&\leq (c-a)\int_a^c (y^{\Delta}(t))^2 \Delta t \\
&\leq -(c-a)\int_a^c y(\sigma(t)) y^{\Delta^2}(t)\Delta t \\
&\leq (c-a)\int_a^c | y(\sigma(t)) || y^{\Delta^2}(t) | \Delta t.
\end{aligned}
\tag{3.14}
$$

Integrating the equation (3.1) from d to t and then taking absolute value for it, we obtain

$$
| \int_d^t y^{\Delta^3}(t)\Delta t | = | \int_d^t \sum_{k=1}^n p_k(s) y(s)\Delta s |, \tag{3.15}
$$

hence employing definition2.1 and triangle inequality, we obtain

$$
| y^{\Delta^2}(t) | = | \int_d^t \sum_{k=1}^n p_k(s) y(s)\Delta s | \leq \int_d^t \sum_{k=1}^n | p_k(s) || y(s) | \Delta s. \tag{3.16}
$$

Substituting inequality (3.16) into (3.14), we have

$$
\begin{aligned}
(y(c))^2 &\leq (c-a)\int_a^c | y(t) | \Delta t \int_d^t \sum_{k=1}^n | p_k(s) || y(s) | \Delta s \\
&\leq (c-a)^2 M^2 \int_a^b \sum_{k=1}^n | p_k(t) | \Delta t.
\end{aligned}
\tag{3.17}
$$

which yields

$$
\int_a^b \sum_{k=1}^n | p_k(t) | \Delta t \geq \frac{1}{(c-a)^2}, \tag{3.18}
$$

then

$$
\frac{1}{(c-a)} \leq \int_a^b \sum_{k=1}^n | p_k(t) | \Delta t < \infty. \tag{3.19}
$$

If $c \leq \sigma(a), T \neq Z$ and $\frac{1}{c-a} \geq \frac{1}{\sigma(a)-a}$, then

$$
\frac{1}{c-a} \geq \max_{a\in T} \frac{1}{\sigma(a)-a} = \infty, \tag{3.20}
$$

we can find that inequality (3.20) is contradiction with formula (3.19).

Remark 3.2 $c \leq \rho(b)$.If and only if $T = Z$, the equal sign holds.

Proof. Let $c \leq d \leq b$, it can be obtained from the above conditions that $y(c) = -\int_c^b y^{\Delta}(t)\Delta t$.The proof method is the same as the Remark1, so it will not be repeated here.

And then we consider the Lyapunov-type inequality of (3.1) on $[a,a']_T = [a,b]_T \cup [b,a']_T$, where $[a,a']_T$. Divide evenly into n parts, that is, $[a,a']_T = \bigcup_{i=1}^n T_i' = \bigcup_{i=0}^{n-1} [t_i', t_{i+1}']$.Let $y(t)$ is a solution of (3.1), and $y(t_i) = 0, y(t) \neq 0 (t \in [a,a']_T, t \neq t_i'), t_0 = a, t_n = a'$.

Theorem 3.2 If $y^{\Delta^2}(t) \neq 0 (t \in [a,b]_T)$ and $y^{\Delta}(b) = y^{\Delta}(a')$,we obtain

$$
\int_a^{a'} \sum_{k=1}^n | p_k(t) | \Delta t > \frac{4n^2}{(a'-a)^2}. \tag{3.21}
$$

Proof. From the mean value theorem in [11], it follows that there exist $c_1, c_2 \in [b, a')$ such that $y^{\Delta^2}(c_1) \le 0 \le y^{\Delta^2}(c_2)$. Then there exist $d \in [c_1, c_2]$ such that $y^{\Delta^2}(d) = 0$. Let

$M = \max_{t\in[a,a']_T} |y(t)| = |y(c)| (c \in [a, a']_T)$, then

$$M = \left|\int_{t_i'}^{c} y^{\Delta}(t)\Delta t\right| \le \int_{t_i'}^{c} |y^{\Delta}(t)|\Delta t, \tag{3.22}$$

$$M = \left|\int_{c}^{t_{i+1}'} y^{\Delta}(t)\Delta t\right| \le \int_{c}^{t_{i+1}'} |y^{\Delta}(t)|\Delta t, \tag{3.23}$$

where $i = 0, 1, \cdots, n-1$. The proof is similar to the proof of Theorem 3.1, and (3.11) is changed into

$$4n^2M^2 \le (a' - a)\int_a^{a'} |y(\sigma(t))||y^{\Delta^2}(t)|\Delta t. \tag{3.24}$$

Since

$$|y^{\Delta^2}(t)| = \left|\int_d^t y^{\Delta^3}(t)\Delta t\right| \le \int_a^{a'} \sum_{k=1}^{n} |p_k(t)||y(t)|\Delta t. \tag{3.25}$$

and substituting (3.25)into (3.24),we have

$$\begin{aligned} 4n^2M^2 &\le (a' - a)\int_a^{a'} |y(\sigma(t))|\Delta t \int_a^{a'} \sum_{k=1}^{n} |p_k(t)||y(t)|\Delta t \\ &< (a' - a)^2 M^2 \int_a^{a'} \sum_{k=1}^{n} |p_k(t)|\Delta t. \end{aligned} \tag{3.26}$$

Dividing the latter inequality of (3.26) by M^2,(3.21) follows immediately.

4. Conclusion

It's obviously that the inequality (3.2) of Theorem 3.1 includes the inequality (2.4) and (3.21) contains (2.5). Although the conditions of Theorem 3.1 and Theorem 3.2 is different from Lemma1.1 and Lemma12, We obtain the conclusions is improvements and extensions for some results of [1] and other relate literature.

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