Algebraic Properties of Star Products 1D m-modal maps

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Abstract

A star product is used to describe a route to chaos by a period-p-tupling , the corresponding metric universalities can be investigated consequently. A recent result is that Feigenbaum scenario will be destroyed by the so-called non-normal star product (NNSP) which does not satisfy an associative law, it is important to distinguish a NNSP and a normal star product (NSP) from a given multiplication table. In this paper, we present the general expression of star products for 1D m-modal maps while some useful lemmas and theorems of Star Products are proofed, these result will provide important clues for the proof of Conjecture 2.1 in section 2.

Keywords

Non-normal star products; Symbolic dynamics.

1. Introduction

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Feigenbaum scenario in unimodal maps arouses interesting researches on metric universalities in dynamical systems [1-8]. Derrida--Gervois—Pomeau [9] has presented an important composition rule of sequences, the so-called star product which can be used to reproduce Feigenbaum numbers by a period-doubling process $(RC)^{*n}$ for unimodal maps, the rigorous algebraic structure of the star product gives all renormalizable maps and their construction scheme of renormalization, while it forms all the equal topological entropy classes in which the Feigenbaum's universalities are contained [10]. Therefore, a series of works are contributed to subsequently. Ringland [11] generated a genealogy of finite kneading sequences by using the hierarchical transformation for the α seed, ψ seeds, and χ seed. Brucks [12] discussed a generalization of the star product transformation to multimodal maps by introducing linear graphs of permutations, which is based on an investigation of the factorization of permutations into products of permutations. Recently, star products of bimodal, trimodal and qudrimodal maps are obtained and the corresponding Feigenbaum metric universalities are generalized hereby [13-16], an interesting result is found that the NSPs will hold the Feigenbaum's metric universalities, the case of destroying Feigenbaum scenario is only appeared by using a NNSP in a bifurcation way $(P_1^n = \underbrace{\cdots X*(X*(\cdots(X*Y)))}_{n-1} = X*P_1^{n-1})$ [17]. Therefore, the study of NNSPs becomes necessary, on the other hand, the number of NNSPs in multimodal maps will go beyond that of NSPs far away. It is known that all star products in unimodal maps and bimodal maps are NSPs, there are six star products in trimodal maps [15], two of them are found NSPs firstly, others are NNSPs. The research of NNSPs will open a way to explore the new scheme of renormalization and universal constants. In this paper, we present an rigorous algebraic expression of star products in 1D m-modal maps systemically, while a conjecture of star products is given according to current progress of star products [13, 15-16, 18].

2. Symbolic dynamics of 1D m-modal maps

In this section, we give the general description of symbolic dynamics of 1D m -modal maps, including coarse-graining description, ordering rule of symbolic sequences, admissibility conditions, the height order relation (HOR), some basic operators and notations. The word-lifting technique is ignored here [19]. Preliminaries in this part is for understanding the expression of star products in section 3.

2.1 Coarse-graining description

First, we give a concise description for symbolic dynamics of 1D m-modal maps. A m-modal map (1) has m critical points denoted as $C_1, C_2, \cdots, C_m$ and m+1 monotone branches as $I_1, I_2, \cdots, I_{m+1}$, respectively (Figure 1).

Let $C_1, C_2, \cdots, C_m$ be the horizontal coordinates of m critical points and $-1 < c_1 < c_2 < \cdots < c_m < 1$ holds. The corresponding iterative system is considered as

$$x_{n+1} = f(x_n, \mu) \tag{1}$$

Here, $f_\mu : I \to I$ is a nonlinear map of endomorphism on the real interval $I = [-1,1]$, $\mu = (c_1, c_2, \cdots, c_m)$. In fact, equation (1) is a general iterative form of 1D m-modal maps while can meet the requirement of the new numerical solution of word-lifting technique [19].

For a given initial point x_0, a numerical orbit $(x_0, x_1, \cdots x_j, \cdots,)$ is obtained by (1), while a sequence $W = w_0 w_1 \cdots w_j \cdots$ is signed by the following coarse-graining or reduction of description [4-5],

$$w_j = \begin{cases} I_1, & for\ x_j \in [-1, c_1), \\ C_1, & for\ x_j = c_1, \\ I_i, & for\ x_j \in (c_{i-1}, c_i), \\ C_i, & for\ x_j = c_i, \\ I_{i+1}, & for\ x_j \in (c_i, c_{i+1}), \\ C_m & for\ x_j = c_m, \\ I_{m+1}, & for\ x_j \in (c_m, 1], \end{cases} \tag{2}$$

where, $i \in Z^+, 2 \le i \le m-1$. Additionally, as the natural order of real numbers on the interval [-1,1], $I_1 < C_1 < I_2 < C_2 < \cdots < I_m < C_m < I_{m+1}$ holds. Let $\mathbf{L} = \{I_i, C_i, I_{m+1}, | \ i \in Z^+, 1 \le i \le m\}$.

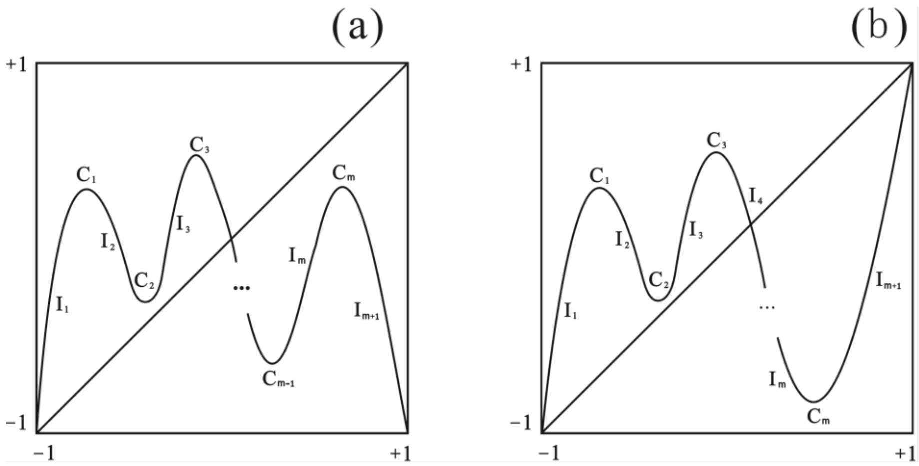

Figure 1: Schematic graphs of m-modal maps with m critical points (for the case of I1 to be an increasing branch): (a) m is odd; (b) m is even.

Let S is a symbolic sequence, a parity operator τ is defined to develop ordering rule of symbolic sequences:

$$\begin{cases} \tau(S)=1, & \text{if the number of monotonically decreasing branch in } S \text{ is even,} \\ \tau(S)=-1, & \text{otherwise;} \end{cases} \tag{3}$$

Ordering rule of sequences: To compare two symbolic sequences S_1 and S_2, let $S_1=\triangle a\cdots$ and $S_2=\triangle b\cdots$, here $\triangle$ is their common leading string and $a\neq b$.

Supposing that $\tau(\triangle)=1$: if $a>b$, then $S_1>S_2$ and vice versa; supposing that $\tau(\triangle)=-1$: if $a>b$, then $S_1<S_2$ and vice versa.

2.2 Admissibility Conditions And Admissible Sets.

For a given symbolic sequence W, $\overline{A}(W)$ is noted as a set of all subsequences which follow letter A in W, where $A\in\mathbf{L}$. The symbolic sequence W is called admissible, if an admissibility condition (4) is satisfied:

$$\begin{cases} \overline{I_i}(W)<\overline{C_i}(W), \overline{I_{i+1}}(W)<\overline{C_i}(W), & \text{if } C_i \text{ is a peak;} \\ \overline{C_i}(W)<\overline{I_i}(W), \overline{C_i}(W)<\overline{I_{i+1}}(W), & \text{if } C_i \text{ is a valley.} \\ i\in Z^+,\ 1\le i\le m. \end{cases} \tag{4}$$

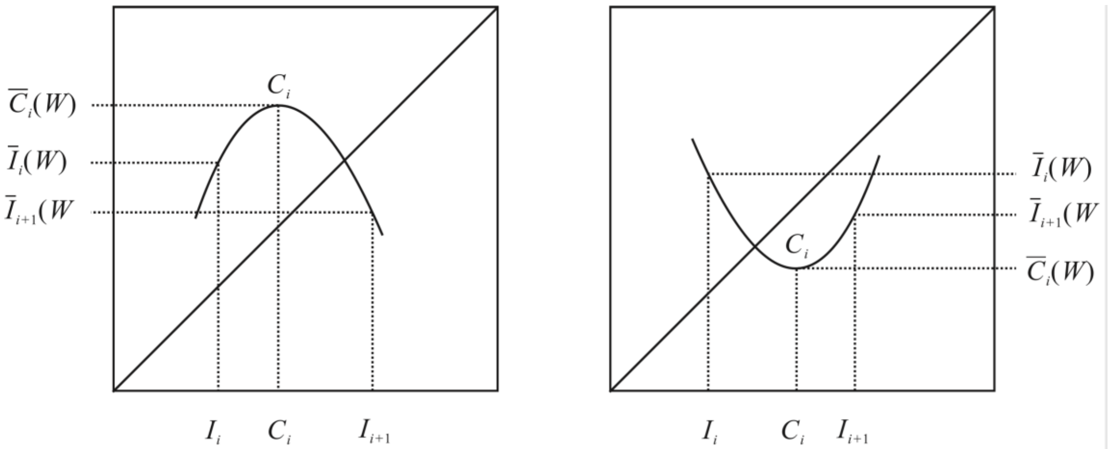

Figure 2: Schematic graphs of admissibility conditions, it shows the relation between letter A and its subsequence $\overline{A}(W)$, here $A\in\mathbf{L}$.

(4) is used to judge whether a symbolic sequence is corresponding to an exist orbit or not, that is to say, whether there has a group of parameters and an initial point x_0, such that W can be reproduced by (1) and (2). It is worth noting that if there is not a critical point in W, e.g. C_k, inequalities including $\overline{C_k}(W)$ in (4) will be invalid and omitted automatically, otherwise valid. For example, if a symbolic sequence W does not pass through any critical point, the non-superstable kneading sequence W is admissible and is called 'flesh' in m -dimensional parameter space [20].

A periodic sequence passing through all m critical points is called a m-tuply superstable kneading }(mSSK) sequence, while it is called a joint point [20]. All admissible symbolic sequences form a set K^0, while all admissible mSSK sequences form a set K^1.

2.3 Height Order Relation (HOR).

The HOR is defined as: Let $1 \le i, j \le m, 1 \le t \le m+1, j \ne i$. Supposing that C_i is a peak, when $\overline{C_i}(W) > \overline{I_t}(W)$ and $\overline{C_i}(W) > \overline{C_j}(W)$ hold simultaneously, we call W has HOR i, else W has no HOR i; supposing that C_i is a valley, when $\overline{C_i}(W) < \overline{I_t}(W)$ and $\overline{C_i}(W) < \overline{C_j}(W)$ hold simultaneously, we call W has HOR i, else has no HOR i. In other words, if W has HOR i, if C_i is a peak, it must be the highest among other peaks; if C_i is a valley, it must be the lowest among other valleys.

The HOR is used in the rule of star products. The HORs in D.G.P. star products [9] and dual star products [13] have not been mentioned, the reason is that they hold already. The HOR is found in trimodal maps and is the inherent property of 1D m-modal maps.

2.4 Star Products of One-dimensional *M*-modal Maps.

From two given admissible words X and Y, a star product will produce a new admissible sequence $X * Y$. Here $Y = y_1 y_2 \cdots y_j \cdots y_{|Y|}, y_j \in \mathbf{L}, |X|$ and $|Y|$ are length of sequence X and Y respectively. A star product $*$ satisfies a multiplication distributive law,

$$X * Y = (X * y_1)(X * y_2)\cdots(X * y_j)\cdots(X * y_{|Y|}) \tag{5}$$

An operator $\bigcup$ of sequence concatenation is introduced to simplify (5),

$$X * Y = \bigcup_{j}^{|Y|}(X * y_j) \tag{6}$$

For 1D m-modal maps, X in (5) must be an admissible mSSK sequence,

we note $X = (X_{k_1}C_{k_1}X_{k_2}C_{k_2}\cdots X_{k_i}C_{k_i}\cdots X_{k_m}C_{k_m})^{\infty} = (\bigcup_{i=1}^{m}(X_{k_i}C_{k_i}))^{\infty} \equiv \bigcup_{i=1}^{m}(X_{k_i}C_{k_i})$, here X_{k_i} is a finite sequence composed of letters from $\{I_t | 1 \le t \le m+1.\}$, while $(k_1, k_2, \cdots, k_m)$ is a permutation of $(1, 2, \cdots, m)$.

$$\begin{pmatrix} 1 & 2 & \cdots & m \\ k_1 & k_2 & \cdots & k_m \end{pmatrix} \tag{7}$$

It shows different orders of critical points being successively passed through in a mSSK sequence. The number of different orders is m!, while it is the number of star products in 1D m-modal maps [15,16,18].

Further in equation (6),

$$X * y_j = \left(\bigcup_{i=1}^{m}\left(X_{k_i}C_{k_i}\right)\right) * y_j = \bigcup_{i=1}^{m}\left(\left(X_{k_i} * y_j\right)\left(C_{k_i} * y_j\right)\right) \tag{8}$$

$$= \bigcup_{i=1}^{m}\left(X_{k_i}C_{k_i}^{\tau\left(X_{k_i}\right)\cdot S\left(C_{k_i} * y_j\right)}\right) \tag{9}$$

$$C_i^{+1} = I_{i+1}, C_i^{-1} = I_i, C_i^0 = C_i \tag{10}$$

The value of $S(C_{k_i} * y_j)$ can be found in the multiplication table [15, 18], here we give its form, where, $S(C_{k_i} * y_j) = \begin{cases} +1 \text{ or } -1, & \text{if } y_j \neq C_{k_i}, \\ 0, & \text{if } y_j = C_{k_i}, \end{cases}$. The method to construct such a multiplication table has been presented [18] and is ignored here.

Table 1 Schematic form of multiplication table

$X_{k_1}C_{k_1}$	$\cdots$	$X_{k_i}C_{k_i}$	$\cdots$	$X_{k_m}C_{k_m}$	y_j
$X_{k_1}C_{k_1}^{\tau(X_{k_1})\cdot S(C_{k_1}*I_1)}$	$\cdots$	$X_{k_i}C_{k_i}^{\tau(X_{k_i})\cdot S(C_{k_i}*I_1)}$	$\cdots$	$X_{k_m}C_{k_m}^{\tau(X_{k_m})\cdot S(C_{k_m}*I_1)}$	I_1
$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$X_{k_1}C_{k_1}^{\tau(X_{k_1})\cdot S(C_{k_1}*C_{k_i})}$	$\cdots$	$X_{k_i}C_{k_i}$	$\cdots$	$X_{k_m}C_{k_m}^{\tau(X_{k_m})\cdot S\left(C_{k_m}*C_{k_i}\right)}$	C_{k_i}
$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$X_{k_1}C_{k_1}^{\tau(X_{k_1})\cdot S(C_{k_1}*I_{m+1})}$	$\cdots$	$X_{k_i}C_{k_i}^{\tau(X_{k_i})\cdot S(C_{k_i}*I_{m+1})}$	$\cdots$	$X_{k_m}C_{k_{m+1}}^{\tau\left(X_{k_{m+1}}\right)\cdot S\left(C_{k_{m+1}}*I_{m+1}\right)}$	I_{m+1}

Conjecture 2.1 *If* $X \in K^1$ and $Y \in K^0(K^1)$, *Y has HOR* k_m, *then* $Z = X * Y \in K^0(K^1)$.

When m takes 2, 3 and 4, the result is verified, no counter-example appears. At present, the proof of conjecture 2.1 is difficult, however, a series of lemmas and theorems presented in section 3 will provide important clues, while one of theorems can tell us whether a star product is NSP or NNSP according to its multiplication table.

3. The Algebraic Properties of Star Products

3.1 A Few Useful Lemmas and Theorems of Star Products

Let $X = \bigcup_{i=1}^{m}(X_{k_i}C_{k_i}) \in K^1$, all corresponding notations are the same as those in section 2.

Condition 3.1 The star product needs that $\tau(I_j) = \tau(X * I_j)$ holds, $j = 1, 2, \cdots, m+1$.

If it is a finite sequence composed of letters I_j, it it is easy to know $\tau(S) = \tau(X * S)$ holds too.

Condition 3.2 Supposing that $u, v \in \mathbf{L}$ and $u < v$, the star product needs that $X * u < X * v$ holds.

Remark 3.3 Conditions 3.1 and 3.2 are the necessary conditions of the star product, in fact, they guide the construction method of multiplication table 1 [18], otherwise conjecture (1) will fail by a counter example.

Lemma 3.4 (i) $X_iC_i^{-\tau(X_i)} < X_iC_i < X_iC_i^{\tau(X_i)}$ holds;

(ii) If C_i is a peak, then $\tau(X_iC_i^{-\tau(X_i)}) = 1$ and $\tau(X_iC_i^{\tau(X_i)}) = -1$ hold;

(iii) If C_i is a valley, then $\tau(X_iC_i^{-\tau(X_i)}) = -1$ and $\tau(X_iC_i^{\tau(X_i)}) = 1$ hold;

(iv) $\tau(\bigcup_{i=1}^{m}(X_iC_i^{\tau(X_i)})) = 1$ holds.

Proof. Supposing that C_i is a peak, then $\tau(C_i^{-1}) = 1$ and $\tau(C_i^{+1}) = -1$ hold apparently. If $\tau(X_i) = 1$, then we have $X_iC_i^{-\tau(X_i)} = X_iI_i$ and $X_iC_i^{\tau(X_i)} = X_iI_{i+1}$, by the ordering rule of symbolic sequences, (i) and (ii) hold; else if $\tau(X_i) = -1$, then we have $X_iC_i^{-\tau(X_i)} = X_iI_{i+1}$ and $X_iC_i^{\tau(X_i)} = X_iI_i$, by the ordering rule of symbolic sequences, (i) and (ii) hold too; For the case that C_i is a valley, (i) and (iii) can be proofed similarly; Supposing that m is odd, the number of peaks p is even, while the number of valleys v is odd, from (ii) and (iii), $\tau(\bigcup_{i=1}^{m}(X_i C_i^{\tau(X_i)})) = \bigcup_{i=1}^{m}(X_i C_i^{\tau(X_i)}) = \underbrace{(-1)\cdots\cdots(-1)}_{p}\cdot\underbrace{1\cdots\cdots1}_{v} = 1$ holds; if m is even, the number of peaks p is even, while the number of valleys v is even, from (ii) and (iii), equation (3.1) holds too.

The proof is over.

Lemma 3.5 In multiplication table 1, let $u, v \in \mathbf{L}$ and $u < C_{k_1} < v$, then $S(C_{k_1} * u) = -1$ and $S(C_{k_1} * v) = 1$.

Proof. $X * u = X_{k_1} C_{k_1}^{\tau(X_{k_1})\cdot S(C_{k_1}*u)}\cdots, X * C_{k_1} = X_{k_1} C_{k_1}\cdots, \quad X * v = X_{k_1} C_{k_1}^{\tau(X_{k_1})\cdot S(C_{k_1}*v)}\cdots,$

from condition 3.2, we have $X * u < X * C_{k_1} < X * v$, by lemma 3.4, $S(C_{k_1} * v) = 1$ and $S(C_{k_1} * u) = -1$ hold apparently.

Corollary 3.6 $\tau(I_j) = \prod_{i=1}^{m} S(C_i * I_j)$ holds, here $j = 1, 2, \cdots, m+1$, $S(C_i * I_j)$ is the content of multiplication table 1.

Proof. From condition 3.1 and lemma 3.4,

$$\tau(I_j) = \tau(X * I_j) = \tau((\bigcup_{i=1}^{m}(X_{k_i} C_{k_i})) * I_j) = \tau(\bigcup_{i=1}^{m}(X_{k_i} C_{k_i}^{\tau(X_{k_i})\cdot S(C_{k_i}*I_j)})$$

$$= (\prod_{i=1}^{m}\tau(X_{k_i} C_{k_i}^{\tau(X_{k_i})}))\cdot\prod_{i=1}^{m} S(C_i * I_j) = \prod_{i=1}^{m} S(C_i * I_j)$$

The proof is over.

Lemma 3.7 For three symbolic sequences S^0, S^1 and S^2, S^0 is composed of letters, $I_i, i = 1, 2, \cdots, m+1$. S^1 and S^2 are composed of letters from set $\mathbf{L}$ and supposing that $S^1 < S^2$ holds, then the following results hold.

(i) If $\tau(S^0) = 1$, then $S^0 S^1 < S^0 S^2$ holds ; (ii) If $\tau(S^0) = -1$, then $S^0 S^1 > S^0 S^2$ holds.

Proof. Let $S^1 = \nabla a \cdots$, $S^2 = \nabla b \cdots$, where ∇ is the common leading string of S^1 and $S^2, a \neq b$.

First, supposing that $\tau(S^0\nabla) = 1$ holds: If $\tau(S^0) = 1$ holds, then $\tau(\nabla) = 1$, so $S^1 < S^2 \Rightarrow a < b \Rightarrow S^0 S^1 < S^0 S^2$; If $\tau(S^0) = -1$ holds, then $\tau(\nabla) = -1$, so $S^1 < S^2 \Rightarrow a > b \Rightarrow S^0 S^1 > S^0 S^2$.

Second, supposing that $\tau(S^0\nabla) = -1$ holds: If $\tau(S^0) = 1$ holds, then $\tau(\nabla) = -1$, so $S^1 < S^2 \Rightarrow a > b \Rightarrow S^0 S^1 < S^0 S^2$; f $\tau(S^0) = -1$ holds, then $\tau(\nabla) = 1$, so $S^1 < S^2 \Rightarrow a < b \Rightarrow S^0 S^1 > S^0 S^2$. The proof is over.

Theorem 3.8 Two symbolic sequences S^1 and S^2 are composed of letters from set $\mathbf{L}$. Supposing that $S^1 < S^2$ holds, then $X * S^1 < X * S^2$ holds.

Proof. Let $S^1 = \nabla a \cdots, S^2 = \nabla b \cdots$, where ∇ is the common leading string of S^1 and $S^2, a \neq b$. We have $X * S^1 = (X * \nabla)(X * a)\cdots$ and $X * S^2 = (X * \nabla)(X * b)\cdots$.

If $\tau(\nabla) = 1$ holds, then $S^1 < S^2 \Rightarrow a < b \Rightarrow X * a < X * b$, from lemma 3.7, $X * S^1 < X * S^2$ holds;

If $\tau(\nabla) = -1$ holds, then $S^1 < S^2 \Rightarrow a > b \Rightarrow X * a > X * b$, from $\tau(X * \nabla) = \tau(\nabla) = -1$ and lemma 3.7, $X * S^1 < X * S^2$ holds.

The proof is over.

Remark 3.9 We call theorem 3.8 as order-preserving theorem of star products, it would be useful to proof conjecture (2.1).

4. Conclusion

In this paper, for the star products of 1D *m*-modal maps, the corresponding algebraic properties are explored and a series of lemmas and a theorem are presented, these rules will play an important role on the proof of the conjecture 2.1. The NNSP will be researched furtherly.

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