Exploration of the Irrational Properties of the Natural Exponent e and pi π and the Product of Sums and Products

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Abstract

The rational or irrational property of real numbers is an arithmetic property, and the rational or irrational characteristics of some important constants are closely related to the properties of integers and the distribution of prime numbers. The natural exponent e and pi π have attracted close attention of mathematicians as two of the most famous irrational numbers, of which the famous Euler formula relates the two. The exploration of the irrational nature of e and π themselves was first proposed by Olds in 1970 and Ivan in 1947, respectively, followed by many scholars who gave different methods of proofs. However, in fact, the systematic discussion and sorting out of the many proofs is not complete, except for the irrational character of the combinatorial numbers (sums, differences, products) of the two, which is also worth studying. This paper is devoted to exploring the construction methods and properties of the relevant auxiliary functions in the process of proving the irrational characterization of e and π and the combined numbers (sum, difference, product) of the two. The present study is divided into three main parts. We first investigate how to construct suitable auxiliary functions for e and π such that the proofs of the irrational properties of both can be described uniformly. We give a key auxiliary function and construct exponential and trigonometric differential equations based on the differences in the properties of e and π, respectively, from which we obtain some inferences that the irrational properties of e and its powerful forms can be proved uniformly, while the power forms of π require individualized construction of specific functions to prove them. Second, we investigate the proof that the sum e+π and the product e·π are irrational numbers, first by using the converse method to prove the existence of at most one rational number for the combination of the four operations of e and π. Subsequently, we discuss the sum and product of algebraic and non-algebraic numbers, pointing out that the subset of algebraic irrational numbers is a pseudo-ring without rational numbers, while the subset of non-algebraic irrational numbers is a pseudo-ring without algebraic rational numbers. Finally, we estimate the irrational properties of e+π and the product e-π based on the linear independence of w-transformations and rational solutions and relate the algebraic closure of rational numbers to the solutions of rational polynomial equations to obtain several important conclusions. It is noteworthy that the study of linear irrelevance of solutions of rational numbers under a system of linear equations of general form is actually an important class of problems in Diophantine analysis, and the proof of the irrational character of e and π and the combined numbers (sums, differences, products) of the two using this line of thought is of high research value.

Keywords

Natural Exponents; pi; Irrational Properties; w-transform; Diophantine Analysis.

1. Introduction

When we talk about pi π, we are usually told that it is an infinite noncyclic decimal number called an irrational number. At the same time, when we study the natural exponent e, we learn that this number is also irrational. Coincidentally both of these numbers are irrational, thus π and e become the two most famous irrational numbers known to people. However, since rational numbers have as many roots as natural numbers, there are far more irrational numbers in the real world than rational numbers, because rational numbers are countable, while irrational numbers are uncountable and have as many roots as all real numbers. Therefore, if we calculate the probability strictly, the probability that a point chosen randomly from the number axis is an irrational number tends to be 100%. In this way, it is not surprising that both π and e are irrational numbers, but these two numbers are very high in mathematics and often appear in various types of functions [1-11]. The mathematician Euler established Euler's formula.

$$e^{i\pi} + 1 = 0 \tag{1.1}$$

In mathematics, Euler's formula is one of the most fascinating formulas that connect several constants that are extremely important and highly characteristic in mathematics. Two transcendental numbers: the natural exponent e, the circumference π; two units: the imaginary unit i, the natural unit 1, and, as is common in mathematics, 0. Because of this, Euler's formula is known as the first formula of the universe.

The rational or irrational nature of real numbers is an arithmetic property, so it is not surprising to encounter important constants [12-17], whose rational or irrational nature is related to the nature of integers and the distribution of primes [1,2], such as the number $\frac{6}{\pi^2} = \prod_{p>2}(1 - \frac{1}{p^2})$.

Moreover, mathematics is an extremely rigorous science, and to assert that both numbers are irrational would require giving proof. $\sqrt{2}$ is the first irrational number ever discovered by man, and the method and procedure of its proof are relatively concise and clear. The proof that π and e are irrational numbers is a bit more complicated. In this paper, we want to find out how to prove that both π and e are irrational numbers and the properties of their combinations (sums, differences, products).

2. Basic Concepts and Preparatory Knowledge

This section describes the basic concepts, notations, and preparatory knowledge used throughout the work.

2.1 Rational and irrational criteria

If $\alpha = \frac{a}{b}$, where $a, b \in Z$ are integers, then the number $\alpha \in R$ is called rational. Otherwise, the number is irrational. Irrational numbers can be classified as algebraic and transcendental numbers. α is algebraic if it is a root of an irreducible polynomial $f(x) \in Z[x]$ with number $\deg(f) > 1$ and vice versa [3].

Lemma 2.1 (Rational Criterion) If a real number $\alpha \in Q$ is a rational number, then there exists a constant $c = c(\alpha)$ such that

$$\frac{c}{q} \leq \left|\alpha - \frac{p}{q}\right| \tag{2.1}$$

holds for any rational fraction $\frac{p}{q} \neq \alpha$. Specifically, if $\alpha = \frac{\alpha}{B}$ then $c \geq \frac{1}{B}$.

This is a mathematical expression about the difficulty of any rational number $\alpha \in Q$ being effectively approximated by other rational numbers [4-6]. On the other hand, the irrational number $\alpha \in R - Q$ can be effectively approximated by rational numbers. If the inequality $\left|\alpha - \frac{p}{q}\right| < \frac{c}{q}$ complementary to Equation 2.1 holds approximately for an infinite number of rational numbers $\frac{p}{q}$, then it is sufficiently clear that the real number $\alpha \in R$ is irrational.

Lemma 2.2 (Irrational Criterion) Let $\psi(x) = o(\frac{1}{x})$ be a monotonically decreasing function, such that $\alpha \in Q$ is a real number, if

$$0 < \left|\alpha - \frac{p}{q}\right| < \psi(q) \tag{2.2}$$

holds for infinitely many rational fractions $p/q \in Q$, then α is irrational [4-6].

Proof: by Lemma 2.1 and assumptions, it follows that

$$\frac{c}{q} \leq \left|\alpha - \frac{p}{q}\right| < \psi(q) = o(\frac{1}{q}) \tag{2.3}$$

However, this is a contradiction because $\frac{c}{q} \neq o(\frac{1}{q})$ A more precise theorem for testing that any real number is irrational is discussed below.

Theorem 2.1 Suppose aR is an irrational number, then there exists an infinite sequence of rational numbers $\frac{p_n}{q}{}_n$ satisfying

$$0 < \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2} \tag{2.4}$$

holds for any integer $n \in N$ [4-6].

For a continuous fraction $a_i \geq a > 1$ of the larger term $\alpha = [a_0, a_1, a_2, \cdots]$, where a is a constant, there is a slightly better inequality.

Theorem 2.2 Let $[a_0, a_1, a_2, \dots]$ be a sequence of continuous fractions $\left\{\frac{p_n}{q_n : n \geq 1}\right\}$ of real numbers that are convergent, then there is.

$$0 < \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{a_n q_n^2} \tag{2.5}$$

holds for any integer $n \in N$ [4-6].

This is a standard mathematical formulation in the literature [4-6], and related proofs appear in similar references [7-9]. A theorem that provides a more general application to almost all real inequalities is as follows.

Theorem 2.3 Let ψ be a monotonically decreasing real function, $\alpha \in R$. If there exists an infinite sequence of rational approximations $\frac{p_n}{q_n}$ such that $\frac{p_n}{q_n} \neq \alpha$ and.

$$0 < \left|\alpha - \frac{p_n}{q_n}\right| < \frac{\psi(q_n)}{q_n} \tag{2.6}$$

and $\sum_q \psi(q) < \infty$ then the real numbers α are approximable to ψ.

2.2 A key helper function

Construct the auxiliary function.

$f(x) = \frac{x^n(1-x)^n}{n!}$, and prove that this function satisfies the following three properties.

Property I $f(x)$ is a polynomial of form $\sum_{i=n}^{2n} \frac{c_i}{i!} x^i$ and satisfies that the coefficients c_i are all integers.

Property II When $0 < x < 1$, $0 < f(x) < \frac{1}{n!}$.

Property III For all integers $m \geq 0$, the m -th order derivatives of $f(x)$ must have integer values at 0 and 1, i.e., $f^{(m)}(0)$ and $f^{(m)}(1)$ are also integers.

Property I and Property II are obviously valid, and Property III is proved below. $f(x)$ is a sum of $n + 1$ terms from the nth power of x to the $2n$th power of x, according to Property I. Therefore, when $m < n$, $f^{(m)}(0)$ is 0, which is of course an integer, and when $m > 2n$, $f^{(m)}(x)$ is constantly 0, which is $f^{(m)}(0)$, of course, also an integer.

And when $n \leq m \leq 2n$, the mth order derivative of $f(x)$ according to the polynomial of property I yields $f^{(m)}(0) = \frac{c_m \cdot m!}{n!}$, and since c_m is an integer and $m \geq n$, this number must be an integer. Therefore $f^{(m)}(0)$ must be an integer. Also, notice that this function has a very obvious symmetry, i.e.

$$f(x) = f(1-x) \tag{2.7}$$

Taking the derivative of order m for both sides of this equation at the same time, and after that we get.

$$f^{(m)}(x) = (-1)^m f^{(m)}(1-x) \tag{2.8}$$

from which we have $f^{(m)}(0) = (-1)^m f^{(m)}(1)$, so since $f^{(m)}(0)$ is an integer, then $f^{(m)}(1)$ is also an integer and Property III holds.

2.3 w -transform

Like the classical Laplace transform, Fourier transform, Merlin transform, finite Fourier transforms, z -transform and other related functional transformation methods transform in the time domain to solve certain problems more only. Similarly, the w -transform transforms some apparently intractable problems in the real domain R into simpler decision problems in the binary domain $F_2 = \{0,1\}$ [10].

Definition 2.1 Let $\alpha \in R$, w -transform be a mapping W:R → $F_2 = \{0，1\}$, defined as follows.

$$w(\alpha) = \lim_{x\to\infty} \frac{1}{2x} \sum_{-x\le n\le x} e^{i\alpha n} \tag{2.9}$$

After normalizing the w -transform, its value can be regressed to π. It can also be modified as needed. w -transform can be a point mapping or an equivalent class of mappings, which is irreversible and often does not require inversion in decision class problem applications.

Lemma 2.3 For any real number $\alpha \in R$, the w -transform satisfies the following conditions.

$$w(2\pi m\alpha) = \begin{cases} 1 \text{ When and only when } \alpha \in Q \\ 0 \text{ When and only when } \alpha \notin Q \end{cases} \tag{2.10}$$

for $m \in Z$.

Proof: Given any rational number $\alpha \in Q$, there exists an integer $m \in Z$ satisfying $\alpha m \in Z$, by definition.

$$w(2\pi m\alpha) = \lim_{x\to\infty} \frac{1}{2x} \sum_{-x\le n\le x} e^{i2\pi m\alpha n} = \lim_{x\to\infty} \frac{1}{2x} \sum_{-x\le n\le x} 1 = 1 \tag{2.11}$$

The above proves that for an integer m, the sequence: $\{2\pi m\alpha : n \in Z\}$ is inhomogeneously distributed. And for any irrational number $\alpha \notin Q$ and integer $m \neq 0$, the sequence: $\{2\pi m\alpha : n \in Z\}$ is uniformly distributed, the proof is the same as Weil's criterion, see Theorem 2.1 of [11].

It can be seen that the function w maps the rational number Q to 1 and the irrational number I=Q $-$ R to 0. The w -transformation derives an equivalence relation on the set of real numbers R=Q $- I$.

A pair of real numbers a and b is equivalent to $a \sim b$ when and only when.

$$w(2\pi a) = w(2\pi b) \tag{2.12}$$

A pair of real numbers a and b is not equivalent to $a \neq b$ when and only when.

$$w(2\pi a) \neq w(2\pi b) \tag{2.13}$$

The next theorem takes the properties of the w -transformation one step further.

Lemma 2.4 For any real number $\alpha \in R$, there exists a map $T: R \to F_3 = \{-1,0,1\}$ which satisfies the following conditions.

$$T(2\pi m\alpha) = \begin{cases} 1 & \text{when and only when } \alpha \text{ is a rational number} \\ 0 & \text{when and only when } \alpha \text{ is a rational number but an algebraic number} \\ -1 & \text{when and only when } \alpha \text{ is rational but not algebraic} \end{cases} \tag{2.14}$$

Lemma 2.5 For any real number $t \neq k\pi$, $k \in Z$, and a sufficiently large integer $x \geq 1$, the rank-sum.

(1)

$$\sum_{-x\le n\le x} e^{i2tn} = \frac{sin((2x+1)t)}{sin(t)} \tag{2.15}$$

(2)

$$\left| \sum_{-x\le n\le x} e^{i2tn} \right| \le \frac{1}{|sin(t)|} \tag{2.16}$$

Proof: Expand this exponential summation into two terms.

$$\sum_{-x\leq n\leq x} e^{i2tn} = e^{-i2t} \sum_{0\leq n\leq x-1} e^{-i2tn} + \sum_{0\leq n\leq x} e^{i2tn} \tag{2.17}$$

Finally, the limit value of the equation can be determined using the geometric series.

3. Proof of key conclusions

3.1 Proof that e is an irrational number

We start with the simplest problem, proving that e is irrational [18-20]. If the function e^x is subjected to a Taylor series expansion at the point $x = 0$, and then after substituting $x = 1$ into the resulting infinite term series expansion, the following well-known formula is obtained, i.e.

$$e = \sum_{i=0}^{\infty} \frac{1}{i!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \ldots\ldots.. \tag{3.1}$$

Without resorting to Taylor series expansions, one can also use the following approach to give a less rigorous proof of the above equation from the definition of e.

We know that by definition, $e = \lim\limits_{n\to\infty}(1 + \frac{1}{n})^n$, and let us first look at the expression $(1 + \frac{1}{n})^n$ for the limit being sought, expanding this power expression according to the binomial decomposition as follows.

$$\begin{gathered}(1 + \frac{1}{n})^n = C_n^0 + C_n^1 \cdot \frac{1}{n} + C_n^2 \cdot \frac{1}{n^2} + C_n^3 \cdot \frac{1}{n^3} + \cdots \ldots. \\ + C_n^n \cdot \frac{1}{n^n} = 1 + \frac{n}{1! \cdot n} + \frac{n(n-1)}{2! \cdot n^2} + \frac{n(n-1)(n-2)}{3! \cdot n^3} + \cdots \ldots. \\ + \frac{n(n-1)(n-2)\ldots\ldots(n-i+1)}{i! \cdot n^i} + \ldots\ldots. \frac{n(n-1)(n-2)\ldots\ldots 2 \cdot 1}{n! \cdot n^n}\end{gathered} \tag{3.2}$$

We observe the i -th term of which (note that i here is independent of n) and set the i -th term to a_i, with

$$\frac{1}{i!} \cdot (\frac{n-i+1}{n})^i < a_i = \frac{n(n-1)(n-2)\ldots\ldots(n-i+1)}{i! \cdot n^i} < \frac{1}{i!} \tag{3.3}$$

It is obvious that both sides of Equation 3.3 converge to $\frac{1}{i!}$ as n tends to $+\infty$, so apply the pinch-force theorem for the limit.

$$\lim_{i\to\infty} a_i = \frac{1}{i!} \tag{3.4}$$

Since $(1 + \frac{1}{n})^n$ expands to an n-term sum, when n tends to $+\infty$, it obviously becomes an infinite term sum. For n tending to $+\infty$, each obtained a_i corresponds to a specific, finite i. As a result of the above derivation, any specific a_i is equal to $\frac{1}{i!}$ and the resulting infinite sum of terms must be,

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots\ldots.. \tag{3.5}$$

Thus.

$$e = \lim_{n\to\infty}(1 + \frac{1}{n})^n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots\ldots. = \sum_{i=0}^{\infty} \frac{1}{i!} \tag{3.6}$$

After proving the above conclusion, the process, and method of proving that e is an irrational number is simpler. We apply the converse method to prove that e is indeed an irrational number.

Assuming that e is a rational number, we may set $e = \frac{a}{b}$, where a and b are positive integers, and we then take a positive integer n and multiply both sides of this equation by $b \cdot n!$ to get.

$$b \cdot n! \cdot e = a \cdot n! \tag{3.7}$$

Obviously, the right side of equation 3.7 is an integer, while its left side is.

$$\begin{gathered} b \cdot n! \cdot e = b \cdot n! \cdot \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} \dots \dots .. \right) \\ = b \cdot n! \cdot (1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} \dots \dots .. + \frac{1}{n!}) + b \cdot (\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} \\ + \frac{1}{(n+1)(n+2)(n+3)} + \dots \dots .) \end{gathered} \tag{3.8}$$

The first term of this equation is clearly an integer, yet the second term is clearly faulty so that the second term is equal to M. We have.

$$\begin{gathered} 0 < M = b \cdot \left(\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots \dots . \right) \\ < b \cdot (\frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \dots \dots .) = \frac{b}{n} \end{gathered} \tag{3.9}$$

Since it is n we arbitrarily choose a positive integer, as long as we get the value of n large enough so that $n > b$, we get $0 < M < 1$, thus making it impossible for M to be an integer. Thus the left side of equation 3.7 is not an integer, while its right side must be an integer, a contradiction.

Thus e cannot be a rational number, and the proof is over.

3.2 Proof that e^k (k is a positive integer) is irrational

If we are familiar with the Taylor series expansion of e^x, we can use the previous method to prove that e is irrational in a similar way to prove that e^2 is also irrational. First assume that $e^2 = \frac{a}{b}$, and then get $b \cdot e = a \cdot e^{-1}$, while using the Taylor series expansion of e and e^{-1}, and find that one of the two sides of the equation is a little larger than some integer and the other side is a little smaller than some integer, in which case the two numbers cannot be equal, thus deriving a contradiction. No further details will be elaborated here [21-24].

Also, thinking a bit more, we can see that studying whether e^k is an irrational number is a relatively meaningful problem. If for any positive integer k, there is for irrational e^k, so that any rational power we obtain is easily irrational, this is because: for any positive rational number $\frac{k}{l}$, both k and l are obviously positive integers and $e^k = (e^{\frac{k}{l}})^l$, but if e^k is irrational, $e^{\frac{k}{l}}$ must also be irrational, because an integer power of a rational number must be rational, not irrational.

As for the negative rational powers of e, it must be the reciprocal of the positive rational powers, because once all the positive rational powers of e are irrational, it is equivalent to proving that its negative rational powers are also irrational.

We assume that there exists some positive integer k. To make e^k a rational number, we can set $e^k = \frac{a}{b}$, and a and b are positive integers. Then we use the auxiliary function $f(x)$ from Section 2.2 to construct a new function satisfying $F(x)$.

$$\begin{gathered} F(x) = k^{2n} f(x) - k^{2n-1} f^{(1)}(x) + k^{2n-2} f^{(2)}(x) + \cdots \dots \\ + (-1)^i k^{2n-i} f^{(i)}(x) + \dots \dots + f^{(2n)}(x) + \dots \dots \end{gathered} \tag{3.10}$$

Since $f(x)$ is a sub-polynomial, the function $F(x)$ is 0 for all terms after the term $f^{(2n)}(x)$, but it does not make a fundamental difference to continue adding up and writing it in the form of an infinite sum of terms. The function $F(x)$ so constructed has a feature that the form of the derivative function is somewhat similar to the original function, so it is easy to calculate to obtain.

$$F'(x) + kF(x) = k^{2n+1} f(x) \tag{3.11}$$

Based on equation 3.11 the differential equation can be constructed as follows.

$$\frac{d}{dx}[e^{kx} \cdot F(x)] = e^{kx} \cdot F'(x) + e^{kx} \cdot k \cdot F(x) = e^{kx} \cdot k^{2n+1} \cdot f(x) \tag{3.12}$$

Thus, we obtain the following integral equation.

$$\begin{aligned} L &= b \cdot \int_0^1 e^{kx} \cdot k^{2n+1} \cdot f(x)\,dx = b \cdot e^{kx} \cdot F(x)|_0^1 \\ &= b \cdot e^k \cdot F(1) - b \cdot F(0) = a \cdot F(1) - b \cdot F(0) \end{aligned} \tag{3.13}$$

According to Property III, $F(1)$ and $F(0)$ are integers, and a and b are also integers, thus L should be an integer. But on the other hand, according to property II, we have.

$$0 < L = b \cdot \int_0^1 e^{kx} \cdot k^{2n+1} \cdot f(x)\,dx < b \cdot e^k \cdot k^{2n+1} \cdot \frac{1}{n!} = \frac{a \cdot k^{2n+1}}{n!} \tag{3.14}$$

At large values of n, $n!$ grows much faster than k^{2n+1}, so it is necessary to choose n large enough so that $n! > a \cdot k^{2n+1}$ and then get $0 < L < 1$, which contradicts that L is an integer. Thus e^k cannot be a rational number and the proof is over.

From the above process, we get the following conclusion: the natural exponent e itself is irrational, while any rational power of e (except 0) is also irrational.

3.3 Proof that π is an irrational number

Proving that π is irrational is a little more complicated than proving that e is irrational. Assuming that π is rational, we can set $\pi = \frac{a}{b}$, where a and b are positive integers, and we define the analogous function $f(x)$ as the following function.

$$g(x) = \frac{x^n(a - bx)^n}{n!} \tag{3.15}$$

and.

$$G(x) = g(x) - g^{(2)}(x) + g^{(4)}(x) + \ldots\ldots + (-1)^n g^{(2n)}(x) \tag{3.16}$$

We can easily find that the polynomial coefficients of $n!\,g(x)$ are all integers and the lowest power of the polynomial is greater than n. On the other hand, according to the proof of Eq. 2.8, it can be deduced that the values of the function $g(x)$ and its derivatives at 0 and $\frac{a}{b}$ are integers, so $G(0)$ and $G(\frac{a}{b})$ are integers. And the constructed function $G(x)$ is related to the function $g(x)$ by the following equation.

$$G''(x) + G(x) = g(x) \tag{3.17}$$

By the elementary integral operation, we can conclude that,

$$\frac{d}{dx}[G'(x)\sin x - G(x)\cos x] = G''(x)\sin x + G(x)\sin x = g(x)\sin x \tag{3.18}$$

Further, we can obtain the following integral equation.

$$L = \int_0^\pi g(x)\sin x\,dx = G'(x)\sin x - G(x)\cos x\,|_0^\pi = G(\pi) + G(0) \tag{3.19}$$

Since $G(0)$ and $G(\frac{a}{b})$ are both integers, but for $0 < x < \pi$, we have.

$$0 < g(x)\sin x < \frac{\pi^n a^n}{n!} \tag{3.20}$$

Therefore, we can choose n large enough so that $n! > \pi^n a^n$, thus making $0 < L < 1$, which contradicts that L is an integer. Thus π cannot be a rational number and the proof is over.

3.4 Proof that π^2 is an irrational number

The properties of π are not as good as e. For the function ex, its derivative is still itself, while π is not, so proving that it is π irrational is a bit more troublesome compared to e. The following is an example of $k = 2$ with the help of function $f(x)$ and its three properties, to prove that π is irrational.

Assume that π^2 is a rational number, you can set $\pi^2 = \frac{a}{b}$, where a and b are positive integers, after using the function $f(x)$ to construct a new function $P(x)$ as follows.

$$P(x) = b^n\left[\pi^{2n}f(x) - \pi^{2n-2}f^{(2)}(x) + \ldots\ldots + (-1)^i\pi^{2n-2i}f^{(2i)}(x) + \ldots..\right] \tag{3.21}$$

$P(x)$ is summed according to the even-order derivatives of $f(x)$, whose second-order derivatives must satisfy the following relation.

$$P''(x) + \pi^2 P(x) = b^n\pi^{2n+2}f(x) \tag{3.22}$$

Unlike the case of proving that e^k is irrational, here the second-order derivatives are involved and Leibniz's law cannot be used directly, but we can use the derivative property of trigonometric functions to construct the following differential equation.

$$\begin{gathered}\frac{d}{dx}[P'(x)\sin\pi x - \pi P(x)\cos\pi x] \\ = P''(x)\sin\pi x + \pi\cos\pi x\cdot P'(x) - \pi\cos\pi x\cdot P'(x) + \pi^2\sin\pi x\cdot P(x) \\ = \sin\pi x\cdot[P''(x) + \pi^2 P(x)] = b^n\pi^{2n+2}\sin\pi x\cdot f(x)\end{gathered} \tag{3.23}$$

Thus, on the one hand.

$$\begin{gathered}L = \frac{1}{\pi}\int_0^1 b^n\pi^{2n+2}\sin\pi x\cdot f(x)\,dx = \frac{1}{\pi}[P'(x)\sin\pi x - \pi P(x)\cos\pi x]\big|_0^1 \\ = P(1) + P(0)\end{gathered} \tag{3.24}$$

Since $b\pi^2 = a$ is an integer, according to Property III of the function $f(x)$, the value of any order derivative of $f(x)$ at 0 and 1 is an integer, so both $P(0)$ and $P(1)$ are integers and L should be an integer.

But on the other hand, according to Property II, we have.

$$\begin{gathered}0 < L = \frac{1}{\pi}\int_0^1 b^n\pi^{2n+2}\sin\pi x\cdot f(x)\,dx = \pi\cdot\int_0^1 (b\pi^2)^n\sin\pi x\cdot f(x)\,dx \\ = \pi\cdot\int_0^1 a^n\sin\pi x\cdot f(x)\,dx < \frac{\pi^n a^n}{n!}\end{gathered} \tag{3.25}$$

In the case that n takes a large value, so that $n! > \pi^n a^n$ and thus $0 < L < 1$, which contradicts that L is an integer. Thus π^2 cannot be a rational number and the proof is over.

Since the derivative of π^x is not itself, we cannot uniformly construct a differential equation similar to e. To prove that π^k is irrational, we need to personalize the construction of the function $P_k(x)$. Therefore, this method cannot uniformly determine whether π^k is irrational or not.

3.5 Proof that the sum $e + \pi$ and the product $e\cdot\pi$ are irrational numbers

In fact, e and π are not only irrational numbers but also transcendental numbers. Of course, proving that they are transcendental numbers is much more complicated than irrational numbers; after all, proving that a number is irrational only requires proving that it is not a root of any one-time integer coefficient equation, but proving that a number is transcendental requires proving that it is not a root of any (no matter how many) integer coefficient equation [25-30]. Also, although we have proved that e and π are transcendental numbers, we do not know whether the combinations like $e \pm \pi$ and $e\cdot\pi$ and e/π are irrational numbers.

The following counterfactual can be used to prove that there is at most one rational number of these four numbers.

(1). If $e + \pi$, $e\cdot\pi$ are rational numbers, then e and π are roots of the equation with rational coefficients.

$$x^2 - (e + \pi)x + e\pi = 0 \tag{3.26}$$

This contradicts the fact that both e and π are transcendental numbers.

(2). If $e+\pi$ and e/π are both rational numbers, then.

$\pi=(e+\pi)/(1+\frac{e}{\pi})$	is a rational number, a contradiction.

Therefore, $e+\pi$, and e/π cannot be rational numbers at the same time.

(3). If both $e+\pi$ and $e-\pi$ are rational numbers, then.

$e=\frac{1}{2}[(e+\pi)+(e-\pi)]$	is a rational number,and	$\pi=\frac{1}{2}[(e+\pi)-(e-\pi)]$	is a rational number, contradictory.

Therefore, $e+\pi$ and $e-\pi$ cannot be rational numbers at the same time.

(4). If $e\cdot\pi$ and e/π are both rational numbers, then.

$\pi^2=(e\cdot\pi)/(e/\pi)$ is a rational number and $e^2=(e\cdot\pi)\cdot(e/\pi)$ is a rational number, a contradiction. Therefore, $e\cdot\pi$ and e/π cannot be rational numbers at the same time.

3.5.1 Sums and products of algebraic and non-algebraic numbers

The algebraic closure of rational numbers consists of all solutions of rational polynomial equations, and the subset of real numbers can be expressed as

$$\bar{Q}=\{\alpha\in R: f(\alpha)=0 \text{ and } f(x)\in Q[x]\} \tag{3.27}$$

Definition 3.1 An irrational number $\alpha\in$ is called an algebraic irrational number when and only when there exists a rational polynomial $f(x)\in Q[x]$, i.e., $f(\alpha)=0$. Otherwise, it is called a non-algebraic irrational number or transcendental number.

Definition 3.2 A subset of algebraic irrational numbers is defined as

$$A=\{\alpha\in R: \alpha \text{ is an irrational number and } f(\alpha)=0\} \tag{3.28}$$

For a rational polynomial $f(x)\in Q[x]$, a subset of A is a proper subset of the set of algebraic integers, which is $A\subset Q$.

Definition 3.3 A subset of a non-algebraic irrational number is defined as

$$T=\{\alpha\in R: \alpha \text{ is an irrational number and } f(\alpha)\neq 0\} \tag{3.29}$$

For any rational polynomial $f(x)\in Q[x]$

Theorem 3.1 The subsets A and T have the following properties.

A subset A of algebraic irrational numbers is a pseudoring without rational number Q. A subset T of non-algebraic irrational numbers is a pseudoring without algebraic rational numbers $\overline{Q}$.

Proof: Take a pair of non-algebraic irrational numbers $\alpha,\beta\in T$ such that $\alpha\beta\notin\overline{Q}$. Then, by Lemma 3.1, the sum $\alpha+\beta\notin T$ and the product $\alpha\beta\in T$ are non-algebraic irrational numbers. This condition $\alpha\beta\notin\overline{Q}$ implies that the subset T does not contain the algebraic rational numbers $\overline{Q}$.

A new subset of numbers is defined in [12], which is a ring without a unit. This subset is a proper subset of the union of algebraic irrational numbers and non-algebraic irrational numbers.

$$\text{S=\{periods\}}\subset A\cup T \tag{3.30}$$

It is easy to prove that the set S is a countable set.

$\alpha,\beta\in R$, and depending on the properties of α and β, the summation $\alpha+\beta$ may be rational, irrational, or transcendental [31-35]. This simple conclusion will be used in the following.

Lemma 3.1 Suppose $\alpha\in R$ is a transcendental number, and $\beta\in R$ is also a real number, then its sum $\alpha+\beta\in R$ is a transcendental number.

Proof: The real numbers $\alpha+\beta$ and $1/\alpha$ are the unique roots of the following polynomial.

$$\begin{aligned} f(x)=\left(x-(\alpha+\beta)\right)\left(x-\frac{1}{\alpha}\right)=x^2-\left(\alpha+\beta+\frac{1}{\alpha}\right)x+1+\frac{\beta}{\alpha} \\ =\frac{1}{\alpha}(\alpha x^2-(\alpha^2+\alpha\beta+1)x+\alpha+\beta) \end{aligned} \tag{3.31}$$

Since $f(x) \in Z[\alpha, \beta][x]$ is a polynomial with transcendental coefficients, it is evident that $\alpha + \beta$ and $1/\alpha$ are not algebraic irrational numbers. Therefore, they are both transcendental numbers.

3.5.2 Linear independence of rational number solutions

Given a set of irrational numbers $\alpha_1, \alpha_2, \cdots, \alpha_d \in R$, the existence of a rational solution $c_1, c_2, \cdots, c_d \in R$ to the linear equation is described as,

$$c_1\alpha_1 + c_2\alpha_2 + \ldots c_d\alpha_d = 0 \tag{3.32}$$

This is an important problem in Diophantine analysis, and a discussion involving the general form of the linear system of equations can be found in the literature [36-39], where the simplest two- or three-parameter cases can be solved given sufficient information about the parameters.

Theorem 3.2 Assume that $\alpha \neq r\pi, r \in Q$ is an irrational number. Then, the following numbers are linearly independent in the range of rational numbers.

$$(1).\ 1\ \alpha\ \pi; \quad (2).\ 1\ \alpha^{-1}\ \pi;$$

Proof: (1). Assuming that these numbers are linearly related in the range of rational numbers Q, consider the equation

$$1 \cdot a + \alpha \cdot b + \pi \cdot c = 0 \tag{3.33}$$

where is $(a, b, c) \neq (0,0,0)$ a rational number solution satisfying Equation 3.33, multiplied by the global least common multiple, and rewritten in equivalent form as

$$2\pi C = -2(\alpha B + A) \tag{3.34}$$

where is the integer $A, B, C \in Z$. To prove that there does not exist any rational number solution to the equation, the two sides are w -transformed to obtain

$$w(2\pi C) = w(-2(\alpha B + A)) \tag{3.35}$$

The left and right sides of equation 3.35 are estimated separately as follows.

Using the characteristic relation $e^{i2\pi C} = 1$, where C is an integer, the w -transformation on the left side of the equation results in

$$w(2\pi C) = \lim_{x\to\infty} \frac{1}{2x} \sum_{-x\le n\le x} e^{i2\pi Cn} = \lim_{x\to\infty} \frac{1}{2x} \sum_{-x\le n\le x} 1 = 1 \tag{3.36}$$

For the irrational number $\alpha B + A$ there is $sin(\alpha B + A) \neq 0$. Using Lemma 2.5 to estimate the right-hand side of Eq. 3.36 as

$$w(-2(\alpha B + A)) = \lim_{x\to\infty} \frac{1}{2x} \sum_{-x\le n\le x} e^{-i2(\alpha B + A)n} \le \lim_{x\to\infty} \frac{1}{2x} \frac{1}{|sin(\alpha B + A)|} = 0 \tag{3.37}$$

The estimates of the w -transform in Eqs. 3.36 and 3.37 contradict Eq. 3.35 as

$$1 = w(2\pi C) \neq w(-2(\alpha B + A)) \tag{3.38}$$

Therefore, there can be no rational number solution $(a, b, c) \neq (0,0,0)$ for equation 3.33, and the proof of (2) is similar.

Theorem 3.3 Let $\alpha \neq r\pi$, an irrational number, then the following numbers are linearly independent in the range of rational numbers.

$$(1).\ 1\ \alpha\ \pi^{-1}; \quad (2).\ 1\ \alpha^{-1}\ \pi^{-1}$$

Proof: (1). Assuming that these numbers are linearly related in the range of rational numbers Q, consider the equation

$$1 \cdot a + \alpha \cdot b + \pi^{-1} \cdot c = 0 \tag{3.39}$$

where $(a, b, c) \neq (0,0,0)$ is a rational number solution satisfying Equation 3.39, transformed into the following equivalent form.

$$2\pi = \frac{-2c}{a + \alpha \cdot b} \tag{3.40}$$

where $a, b, c \in Q$ is a rational number, and to prove that there does not exist any rational number solution to equation 3.39, the w-transformation of its two sides yields

$$w(2\pi) = w(\frac{-2c}{a + \alpha \cdot b}) \tag{3.41}$$

The estimates for each of the left and right sides of equation 3.41 are as follows.

Using the characteristic relation $e^{i2\pi} = 1$, the result of the w -transformation of the left side of the equation is

$$w(2\pi) = \lim_{x\to\infty} \frac{1}{2x} \sum_{-x\le n\le x} e^{i2\pi n} = \lim_{x\to\infty} \frac{1}{2x} \sum_{-x\le n\le x} 1 = 1 \tag{3.42}$$

For an irrational number α with $sin(\frac{-2c}{a+\alpha\cdot b}) \neq 0$, using Lemma 2.5 to estimate the right-hand side of Eq. 3.41 as

$$w(\frac{-2c}{a + \alpha \cdot b}) = \lim_{x\to\infty} \frac{1}{2x} \sum_{-x\le n\le x} e^{i2(\frac{-2c}{a+\alpha\cdot b})n} = \lim_{x\to\infty} \frac{1}{2x} \frac{1}{\left|sin(\frac{-c}{a + \alpha \cdot b})\right|} = 0 \tag{3.43}$$

The estimation results of the w -transform in Eqs. 3.42 and 3.43 contradict Eq. 3.41 as

$$1 = w(2\pi) \neq w(\frac{-2c}{a + \alpha \cdot b}) \tag{3.44}$$

Therefore, there can be no rational number solution $(a, b, c) \neq (0,0,0)$ for equation 3.39, and the proof of (2) is similar to this.

3.5.3 Proof that the sum $e + \pi$ and the product $e\pi$ are irrational numbers

Based on the w -transformation it can be argued that the number $ln\,\pi$ is irrational. To confirm this, assume that it is a rational number $ln\,\pi = r \in Q$ and consider the equivalence equation

$$2\pi = 2e^r \tag{3.45}$$

By the Hermite-Lindemann theorem, e^r is transcendental, and the proof is given in Eqs. 3.10 to 3.14. Taking the w -transformation for both sides of Eq. 3.45, we get

$$w(2\pi) = w(2e^r) \tag{3.46}$$

Using the characteristic relation $e^{i2\pi} = 1$, an estimate for the left-hand side of Eq. 3.46 yields

$$w(2\pi) = \lim_{x\to\infty} \frac{1}{2x} \sum_{-x\le n\le x} e^{i2\pi n} = \lim_{x\to\infty} \frac{1}{2x} \sum_{-x\le n\le x} 1 = 1 \tag{3.47}$$

For the irrational number e^r with $sin(e^r) \neq 0$, using Lemma 2.5 to estimate the right-hand side of Eq. 3.46 gives

$$w(2e^r) = \lim_{x\to\infty} \frac{1}{2x} \sum_{-x\le n\le x} e^{i2e^r n} \le \lim_{x\to\infty} \frac{1}{2x} \frac{1}{|sin(e^r)|} = 0 \tag{3.48}$$

The estimates of the w -transform in Eqs. 3.47 and 3.48 contradict Eq. 3.46, specifically with

$$1 = w(2\pi) \neq w(2e^r) = 0 \tag{3.49}$$

Therefore, $_{\ln}\pi \in R$ is not a rational number.

A similar analysis applies to other numbers such as e^r, but additional work is needed to prove the irrationality of $log_2\,\pi$ and 2^π.

Lemma 3.2 The summation $e + \pi$ and the product $e\pi$ of e and π are irrational numbers

Proof: (1). It follows from Theorem 3.2 that the equation $1 \cdot a + e \cdot b + \pi \cdot c = 0$ has no rational number solution $(a, b, c) = (0, b, c) \neq (0,0,0)$. Therefore $e + \pi = r_0$ has no solution $r_0 \in Q$.

(2). From Theorem 3.3, we know that the equation $1 \cdot a + e \cdot b + \pi^{-1} \cdot c = 0$ has no rational number solution $(a, b, c) = (0, b, c) \neq (0,0,0)$. Therefore $e = r_1\pi^{-1}$ has no rational solution $r_1 \in Q$.

Lemma 3.3 Both $e + \pi$ and $e\pi$ are transcendental numbers (non-algebraic irrational numbers).

Proof: (1) The irrational numbers $e+\pi$ and $e\pi$ are the unique roots of the following polynomials.

$$\begin{aligned} f(x) &= \big(x-(e+\pi)\big)(x-e^{-1}\pi) \\ &= x^2-(e+\pi+e^{-1}\pi)x+\pi+e^{-1}\pi^2 \\ &= \frac{1}{e}(ex^2-(e^2+e\pi+\pi)x+e\pi+\pi^2) \end{aligned} \tag{3.50}$$

Since $f(x)\in Z[e,\pi][x]$ is a polynomial with transcendental coefficients, it follows that $e+\pi$ and $e^{-1}\pi$ are not algebraic irrational numbers. Therefore, they are transcendental numbers (roots of non-algebraic polynomials).

(2) The irrational numbers $e^{-1}+\pi$ and $e\pi$ are the unique roots of the following polynomials.

$$\begin{aligned} g(x) &= \big(x-(e^{-1}+\pi)\big)(x-e\pi) \\ &= x^2-(e^{-1}+\pi+e\pi)x+\pi+e^{-1}\pi^2 \\ &= \frac{1}{e}(ex^2-(1+e^2\pi+e\pi)x+e\pi+\pi^2) \end{aligned} \tag{3.51}$$

Since $g(x)\in Z[e,\pi][x]$ is a polynomial with transcendental coefficients, it follows that $e^{-1}+\pi$ and $e\pi$ are not algebraic irrational numbers. Therefore, they are transcendental numbers (roots of non-algebraic polynomials).

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